BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

EARLE R. HEDRICK

DAVID R. CURTISS WILLIAM R. LONGLEY

WITH THE ASSISTANCE OF

HAROLD T. DAVIS HARRY W. KUHN

THEOPHIL H. HILDEBRANDT DERRICK N. LEHMER

JOHN R. KLINE HENRY L. RIETZ

CAROLINE E. SEELY

VOLUME XXXVIII, NUMBER 8

AUGUST, 1932

PUBLISHED BY THE SOCIETY

MENASHA, WIS., AND NEW YORK

1932

PUBLISHED MONTHLY

Whole No. 395 $9.00 a Year

Entered as second class matter April 24, 1926 at the post office at Menasha, Wis., under the act of August 24, 1912.

AMERICAN MATHEMATICAL SOCIETY

Mail Address: 501 West 116th Street, New York, N.Y., for the Secretary, Treasurer, Librarian, and Clerk.

Office: Room 413, Physics Building, Columbia University, New York City.

President: Dean L. P. Eisenhart, Princeton University, Princeton, N.J.

Secretary: Dean R. G. D. Richardson, Brown University, Providence, R.I.

Associate Secretaries: Professor Tomlinson Fort, Lehigh University, Bethlehem, Pa.; Professor M. H. Ingraham, University of Wisconsin, Madison, Wis.; Dean T. M. Putnam, University of California, Berkeley, Calif.

Treasurer: Professor G. W. Mullins, Barnard College in Columbia University, New York City.

Librarian: Professor R. C. Archibald, Brown University, Providence, R.I.

Clerk: Dr. Caroline E. Seely, 413 Physics Building, Columbia University, New York City.

Editor-in-Chief of the Bulletin: Professor E. R. Hedrick, University of California at Los Angeles, Los Angeles, Calif.

Managing Editor of the Transactions: Professor R. D. Carmichael, University of Illinois, Urbana, Ill.

Chairman, Colloquium Editorial Committee: Professor R. L. Moore, University of Texas, Austin, Tex.

Managing Editor of the American Journal of Mathematics: Professor Abraham Cohen, Johns Hopkins University, Baltimore, Md.

Representatives of the Society in the Division of Physical Sciences of the National Research Council: Professors James Pierpont, D. R. Curtiss, and E. R. Hedrick.

Representatives on the Council of the American Association for the Advancement of Science: Professors Louis Ingold and M. H. Ingraham.

Society Visiting Lecturer for 1931-32: Professor R. L. Moore, University of Texas, Austin, Tex.

Colloquium Speaker for 1932: Professor J. F. Ritt, Columbia University, New York City.

Chairman, Committee on Arrangements for the Semi-Centennial: Professor T. S. Fiske, Columbia University, New York City.

SUMMATION OF FOURIER SERIES*

BY EINAR HILLE

1. *Introduction.* My topic, the summation of Fourier series, has been dealt with in addresses delivered to this Society by C. N. Moore, G. H. Hardy, and again by C. N. Moore, the last one less than two years old.† If I venture to speak on the same subject again, it is because the field is in a state of brisk and steady development so that it is possible for me to deal with matters which have not been exhausted by previous speakers. In particular, I am happy to be able to include in my report a number of results, published and unpublished, found by J. D. Tamarkin and myself during the last few years.

The term Fourier series is used in at least five different senses in the current literature. In the present report the term signifies a trigonometrical Fourier-Lebesgue series, that is, a series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{1}$$

or, in the complex form preferred nowadays,

$$\sum_{n=-\infty}^{+\infty} f_n e^{nix} \tag{2}$$

where the coefficients are determined by

$$\left.\begin{matrix} a_n \\ b_n \end{matrix}\right\} = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\left\{\begin{matrix} \cos \\ \sin \end{matrix}\right\} nt\, dt, \quad f_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t)e^{-nit}dt \tag{3}$$

respectively. Here $f(x)$ is supposed to be integrable in the sense of Lebesgue in the interval $(-\pi, +\pi)$, and is extended by the convention $f(x+2\pi)=f(x)$ outside this interval.

---

* An address read by invitation of the program committee before the Society at New York, March 25, 1932, as part of a symposium on summability.

† The addresses of C. N. Moore have appeared in this Bulletin, vol. 25 (1918–19), pp. 258–276, and vol. 37 (1931), pp. 240–250.

These formulas associate a formal series with every integrable function $f(x)$, the *Fourier series* of $f(x)$. It is natural to ask in what sense the series represents or determines the function. If we write

$$(4) \quad s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) = \sum_{k=-n}^{+n} f_k e^{kix},$$

our first guess is of course that

$$(5) \quad \lim_{n\to\infty} s_n(x) = f(x).$$

All the evidence accumulated during a century indicates, however, that this relation must be considered as an exception rather than as the rule, though a quantitative measure of the improbability of (5) is still lacking.

Ordinary convergence being a clear failure in the study of Fourier series, it is obviously necessary to consider some notion of generalized convergence instead. Essentially two different lines of attack are available here. We may use the modern theory of systems of integrable functions involving the concepts of strong convergence, weak convergence, convergence in measure and so on. Or we may use the theory of transformations of sequences together with the related theory of singular integrals. It is with the second mode of procedure that I shall be concerned here. I want to emphasize, however, that these two methods are closely interwoven, and that both are indispensable for a complete theory of Fourier series.

The early history of summation of Fourier series has been outlined often enough. Let me merely recall here that the theory received its first strong impetus from the fundamental paper of Fejér in 1903 [11*; preliminary communications go back to 1900] which was followed by the investigations of Lebesgue in 1905–09 [26, 27, 28]. These memoirs form the point of departure of the modern theory of Fourier series. They also kindled a wide interest among analysts in the theory of summability which had already proved its usefulness in connection with the problem of analytic continuation around 1900 [see, for example, 3, 29, 30, 31].

* Reference to the Bibliography at the end of this paper.

2. *Effectiveness.* Throughout the development of the theory of summation of Fourier series we may discern two different points of view, a local one and a global one, which occasionally are in conflict but usually complete each other.

The phenomena of convergence or summability of a Fourier series are of a purely local character. Only the properties of the function in the immediate neighborhood of a point determine whether or not its Fourier series is summable at that point by a particular definition of summability. We should like to know what local properties of the function imply such summability of its Fourier series, and, conversely, what properties of the function are implied by its Fourier series being so summable. We have here two types of limiting processes, one referring to the function, the other referring to its Fourier series, and the question is when the existence of one limit implies that of the other.

This is the local problem, but there is also a problem in the large. Every function $f(x) \subset L$ determines a unique Fourier series, but to a given Fourier series correspond infinitely many functions the difference of any two of which is a null function. One of the main problems of our theory is how to pass from the series to the function, or, rather, to any one of the set of equivalent functions. This requires that we be able to assign a sum to the series for almost all values of x, and that the sum-function has the given series as its own Fourier series. The global point of view thus leads to the question: *Does a given definition of summability sum Fourier series to "the correct sum" almost everywhere?* The characterization of all such definitions then becomes a problem of interest and importance both to the theory of Fourier series and to the theory of summability.

But one can also occupy a more general point of view of which those already mentioned are only special aspects. Let there be given a class C of series

$$\sum_{-\infty}^{+\infty} c_n u_n(x), \tag{6}$$

the terms of which are defined for $a \leqq x \leqq b$. Suppose further that with each series of C there is associated a function $f(x)$ defined for almost all x in (a, b). To fix our ideas, we may think of the class L of all Fourier-Lebesgue series, $f(x)$ being in this

case one of the equivalent functions which give rise to the particular series. But we could also let C be the class of conjugate series, or of Fourier-Denjoy series, or of Fourier series in the sense in which this term is used in the theory of almost periodic functions, or of Legendre series etc. In each case it is possible to make the required association between series and function. With each such function $f(x)$ we associate a set E_f of points in (a, b) where $f(x)$ has a finite definite value and satisfies a prescribed condition of regularity. The latter condition will usually imply that $f(x)$ and (or) some related function are continuous in some sense or other. We can think of ordinary continuity or continuity in the mean or strong continuity (a Lipschitz condition or a similar one). Finally, let a definition of summability be given, A say, which associates with the series (6) a sequence or a one-parameter family of functions $T(x, \omega; f(\cdot), A)$. If now

$$\lim_{\omega \to \infty} T(x, \omega; f(\cdot), A) = f(x) \tag{7}$$

for every series in C and for every x in E_f, we say that *the definition A is (C, E_f)-effective*. The notation indicates that this definition can be used for the summing of any series in C, and will assign as the sum of the series the associated function $f(x)$ for every $x \subset E_f$. It should be realized that E_f may be vacuous and that we disregard entirely what happens in the complementary set $(a, b) - E_f$.

In (7) we presuppose convergence in the sense of Cauchy, but we can reach still greater generality by considering various forms of generalized convergence, e.g., convergence in the mean, in which case the definitions have to be modified in an obvious manner. (For the development of this terminology and this point of view, see Hille and Tamarkin [20], third and fourth notes, [21] and [22].) A few examples will clarify the notation.

(i) Let C be the class L of all Fourier-Lebesgue series, $f(x)$ any one of the equivalent functions giving rise to the series. Let E_f be the set of points where $f(x)$ is continuous. We say that A is *(F)-effective* if it has this type of (C, E_f)-effectiveness.

(ii) Let C and $f(x)$ be defined as above and let E_f be the set of points where

$$\lim_{h \to 0} \frac{1}{2h} \int_{-h}^{+h} | f(x + t) - f(x) | \, dt = 0. \tag{8}$$

We designate this type as *(L)-effectiveness*.

Thus A is (F)-effective if it sums the Fourier series of $f(x)$ to the sum $f(x)$ at all points of continuity of $f(x)$, and (L)-effectiveness implies summability to the sum $f(x)$ almost everywhere. It is clear that every (L)-effective definition is also (F)-effective, but the converse is scarcely likely to be true. There exist definitions of summability which are (F)-effective, and which have not been proved to be (L)-effective so far.

In the following I shall be concerned chiefly with the problem of what definitions of summation are (F)- or (L)-effective. Before leaving these general considerations I want to emphasize, however, that this is only a particular case of the general question of (C, E_f)-effectiveness, though one of the most important cases. It is customary to consider various associated series simultaneously with the theory of Fourier series proper such as *the conjugate* (=allied) *series, the derived series of functions of bounded variation* and their *conjugate series.* Without attempting to be systematic, I shall frequently mention what is known about the corresponding effectiveness problems for such classes of series.

3. *Conditions of Effectiveness.* We pass over to the question of how to decide if a given definition of summation A is (F)- or (L)-effective. A is supposed to associate with $f(x) \subset L$ a family of functionals $T(x, \omega; f(\cdot), A)$. A particularly simple example is that in which A is a regular linear sequence-to-sequence transformation with finite reference defined by a triangular matrix $\|a_{m,n}\|$ and the equations

$$y_m = \sum_{n=0}^{m} a_{mn}x_n, \qquad (m = 0, 1, 2, \cdots). \tag{9}$$

Substituting

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x+t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt \tag{10}$$

for x_n, we get

$$T(x, m; f(\cdot), A) = \int_{-\pi}^{+\pi} f(x+t)K_m(t)dt, \tag{11}$$

where

$$K_m(t) = \sum_{n=0}^{m} a_{mn} \frac{\sin (n + \frac{1}{2})t}{2\pi \sin \frac{1}{2}t}. \tag{12}$$

A necessary and sufficient condition that

$$T(x, m; f(\cdot), A) \to f(x) \text{ as } m \to \infty \tag{13}$$

at every point of continuity of $f(x)$ is given by

$$\int_0^\pi | K_m(t) | \, dt < M \tag{14}$$

for a fixed finite M independent of m. The representation (11) and condition (14) remain valid under much more general assumptions on A. But if we merely assume that $T(x, \omega; f(\cdot), A)$ is a linear functional of $f(\cdot)$ which as a function of x belongs to L for every finite ω and which converges to $f(x)$ at every point of continuity as $\omega \to \infty$, then the integrals in (11) and (14) have to be replaced by suitable Stieltjes integrals.

No necessary and sufficient conditions for (L)-effectiveness seem to be known. For a rather wide class of definitions, including all regular definitions of finite reference, a set of sufficient conditions is given by (14) together with

$$\lim_{m \to \infty} K_m(t) = 0, \quad t \neq 0, \tag{15}$$

$$\int_0^\delta | \, d_t[tK_m(t)] \, | < M, \quad \delta < \delta_0, \; m_0 < m. \tag{16}$$

It should be mentioned, however, that condition (16) is apt to fail in all except the very simplest cases occurring in the theory of Fourier series.

Our problem can also be approached through *the relations of relative inclusion* known to exist between various definitions of summability. If every series summable A_2 is also summable A_1 to the same sum, we say, following W. A. Hurwitz, that the definition A_1 includes the definition A_2. Hence, *if A_1 includes A_2 and if A_2 is (C, E_f)-effective, so is A_1.* Now Cesàro's definition of order $\alpha > 0$ is known to be (L)-effective (G. H. Hardy [16]; many other proofs are known). It follows that *any definition A which includes a Cesàro definition of positive order is necessarily (L)-effective.* Incidentally, such a definition will also sum the

conjugate series, the derived series of a function of bounded variation and its conjugate series, each to its proper sum for almost all values of x. The value of this and of similar criteria is limited by the fact that (L)-effective definitions are not necessarily comparable or even consistent, nevertheless such criteria will frequently be used in the discussion below.

4. *Plan of Survey.* I proceed to the main part of this address which is devoted to a preliminary census of the known definitions of summability from the point of view of (F)- and (L)-effectiveness. In this short exposition I can obviously only consider some of the more important definitions where the actual choice of course is largely dictated by my own temporary interests. The classification of definitions of summability is not far developed. I use the customary grouping according to finite or infinite reference; a third heading is added for summation of integrals.

I. Definitions with Finite Reference

I consider four separate types of definitions of summability based upon a single generating sequence (§§5–8) and a fifth one based upon two generating sequences (§9). The discussion of means of the closed cycle which may be of finite or infinite reference starts in §10 and is continued in §11 of II. All these six types contain the definitions of Cesàro as special cases.

5. *Retrogressive Means.* We employ a sequence of complex numbers $\{p_\nu\}$ such that

$$P_n \equiv p_0 + p_1 + \cdots + p_n \neq 0, \tag{17}$$

and take as the generalized limit of the sequence $\{s_n\}$ the expression

$$(N, p_\nu) - \lim s_n = \lim_{n\to\infty} P_n^{-1}(p_n s_0 + p_{n-1}s_1 + \cdots + p_0 s_n) \tag{18}$$

if this limit exists. The conditions for regularity are

$$\sum_{k=0}^{n} |p_k| < C|P_n|, \quad p_n/P_n \to 0. \tag{19}$$

It is customary to refer to these means as those of Nörlund [36] though the first discovery goes back to Woronoi [54]. The term

"retrogressive" is used here for the purpose of comparison with the means of §6, and refers to the manner in which the weights $\{p_\nu\}$ are attached to the elements of the sequence $\{s_n\}$.

The relative inclusion theory of these means has been studied by M. Riesz [43]. The condition that (N, p_ν) shall include $(N, 1) = (C, 1)$ is particularly simple, viz.

$$n|p_0| + \sum_{k=1}^{n}(n-k+1)|p_k - p_{k-1}| < C|P_n|, \tag{20}$$

and is satisfied, for example, if p_n is positive and monotone increasing. Any such definition (N, p_ν) is consequently (L)-effective. A direct study of the effectiveness of these means with respect to Fourier series has been made by Hille and Tamarkin ([20], first note, [21]) who found that the following conditions are sufficient for (L)-effectiveness:

$$n|p_n| < C|P_n|, \tag{21}$$

$$\sum_{k=1}^{n} k|p_k - p_{k-1}| < C|P_n|, \tag{22}$$

$$\sum_{k=1}^{n} \frac{|P_k|}{k} < C|P_n|. \tag{23}$$

If these conditions are satisfied the method (N, p_ν) will also sum the conjugate series, the derived series of a function of bounded variation, and its conjugate series to the proper sums almost everywhere. If $p_n > 0$ and satisfies (21) and (22), thus in particular if p_n is ultimately monotone decreasing, then (23) is a necessary as well as sufficient condition for (F)-effectiveness. A simple example of a definition which is not (F)-effective is given by the harmonic mean, $(N, (\nu+1)^{-1})$.

6. *Progressive Means.* We again employ a sequence $\{p_\nu\}$ satisfying (17), but the generalized limit is taken to be

$$(R, p_\nu) - \lim s_n = \lim_{n\to\infty} P_n^{-1}(p_0 s_0 + p_1 s_1 + \cdots + p_n s_n) \tag{24}$$

instead. The conditions of regularity are now

$$\sum_{k=0}^{n}|p_k| < C|P_n|, \quad P_n \to \infty. \tag{25}$$

We can refer to these weighted means as progressive means. If $p_n>0$ they become a particular case of the discrete Riesz means $R(P_n, 1)$. The choice $p_\nu=1$ gives $(R, 1)=R(n, 1)=(C, 1)$. Except for particular instances, the question of effectiveness of such means does not seem to have been studied, but there is plenty of information available from the inclusion theory. Thus G. H. Hardy [15] has proved that any series summable (R, p_ν), $p_\nu>0$, is summable (R, p_ν, ϵ_ν) to the same sum if the latter definition is regular and ϵ_ν is monotone decreasing. It follows in particular that (R, ϵ_ν), where $\sum\epsilon_\nu$ is divergent, is always an (L)-effective definition. As a special case we note that the logarithmic mean $(R, (\nu+1)^{-1})$ is (L)-effective whereas we have seen that the harmonic mean $(N, (\nu+1)^{-1})$ is not even (F)-effective. (For a recent study of the application of the logarithmic mean to Fourier series, see Hardy [17].) Necessary and sufficient conditions that (R, p_ν) include $(R, 1)=(C, 1)$ are simply conditions (21) and (22) of §5, which are satisfied if, e.g., p_n is a monotone increasing function of n which does not grow faster than every power of n. Any such mean is (L)-effective.

7. *Typical Means.* In order to be in agreement with standard notation we put $P_n=\lambda_{n+1}$, and suppose $p_n=\lambda_{n+1}-\lambda_n>0$, $n\geqq 0$. The typical means of M. Riesz [18, 41] of type λ, order κ, of the first kind, (R, λ, κ), involve taking as the generalized sum of the series $\sum u_n$ the expression

$$\lim_{\omega\to\infty} \sum_{\lambda_n<\omega} \left[1-\frac{\lambda_n}{\omega}\right]^\kappa u_n \tag{26}$$

when this limit exists. Here ω is a continuous parameter. If, however, $\omega\to\infty$ through a denumerable sequence of values, usually through the set $\{\lambda_n\}$ itself, we get various forms of discrete Riesz means, $R(\lambda, \kappa)$. We use the latter notation exclusively for the case in which ω is restricted to the set $\{\lambda_n\}$. We have already observed that $R(\lambda, 1)=(R, \lambda_{\nu+1}-\lambda_\nu)$ and it is easy to see that $R(n, \kappa)=(N, (\nu+1)^\kappa-\nu^\kappa)$. Since every series summable (R, n, κ) is also summable (R, λ, κ) to the same sum if λ_n is a logarithmico-exponential function of n which does not tend to infinity faster than every power of n [18], it follows that any such definition (R, λ, κ) is (L)-effective.

Applications of various types of typical means to the theory

of Fourier series are due to, among others, S. Chapman [7, 8], M. Riesz [41], M. H. Stone [48], and W. H. Young [55, 56].

8. *Momental Means.* We designate the generating sequence by $\{\mu_n\}$. The μ's are supposed to be moments of a mass distribution over the interval (0, 1). To be precise, we assume the existence of a function $q(u)$ with the following properties:

(27) $q(u)$ *is of bounded variation in* (0, 1);

(28) $q(u)$ *is continuous at* $u=0$ *and* $q(0)=0$;

(29) $q(1)=1$;

$$\mu_n = \int_0^1 u^n d[q(u)], \qquad (n = 0, 1, 2, \cdots). \tag{30}$$

The generalized limit of the sequence $\{s_n\}$ is taken to be

$$\lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} s_k \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} \mu_{j+k} \tag{31}$$

if this limit exists. According to Hausdorff [19] every such function $q(u)$ gives rise to a regular definition of summability which we denote by $[H, q(u)]$. All such definitions are known to be mutually consistent, and Cesàro's definitions of positive order belong to the set. The *analytically regular definitions* of Hurwitz and Silverman [23] also belong to this class.

The theory of relative inclusion of momental means offers a fascinating study; particular results found by Hausdorff [19] and by Hille and Tamarkin [unpublished] throw an interesting light over the question of the effectiveness of these means with respect to Fourier series. A direct attack on the latter question has been made by Hille and Tamarkin [20, details unpublished] who have shown that the properties of the Fourier transforms of $q(u)$ are decisive for this problem. Put

$$C(v) = \int_0^1 [q(u) - u] \cos uv \, du, \tag{32}$$

$$J(\epsilon) = \int_\epsilon^1 \frac{Q(u)}{u} du + \int_0^{1-\epsilon} \frac{Q(1) - Q(u)}{1-u} du, \tag{33}$$

$$Q(u) = \int_0^u |\, dq(s)\,| \,.$$

The condition

$$\int_0^\infty |C(v)|\,dv < \infty \tag{34}$$

is necessary for the (F)-effectiveness of the definition $[H, q(u)]$; and (34) together with

$$\lim_{\epsilon \to 0} J(\epsilon) < \infty \tag{35}$$

is sufficient.

Condition (34) is of a type which we shall encounter again in §§10 and 11. It expresses the fact that *the Fourier cosine transform of a certain kernel is integrable over the range* $(-\infty, +\infty)$; it imposes restrictions on the kernel analogous to those required for *the absolute convergence of a Fourier series*. Pursuing this analogy, Hille and Tamarkin [20, third note] have announced a number of conditions, necessary and sufficient or merely sufficient, that the Fourier transform $G(u)$ of a function $g(u)$ shall have this property. Among these conditions the following may be emphasized:*

1. $G(u) \subset \mathfrak{L}_1$, *if* (i) $g(u) \subset \mathfrak{L}_p$ *for some* p, $1 < p \leqq 2$, (ii) $g(u)$ *is of bounded variation in* $(-\infty, +\infty)$, *and* (iii) $g(u)$ *is continuous in* $(-\infty, +\infty)$, *its modulus of continuity,* $\omega(h)$, *satisfying the condition*

$$\int_0^1 [\omega(h)]^{1/p'} h^{-1} dh < \infty. \tag{36}$$

2. $G(u) \subset \mathfrak{L}_1$, *if* (i) $g(u) \subset \mathfrak{L}_p$ *for some* p, $1 < p \leqq 2$, (ii) $g(u)$ *is absolutely continuous and* $g'(u) \subset \mathfrak{L}_1$, *and* (iii)

$$\int_0^1 \Omega(h) h^{-1} dh < \infty,$$

* We denote by $\mathfrak{L}_p$, $1 \leqq p \leqq 2$, the class of functions which are measurable and the pth power of the absolute values of which are integrable over the interval $(-\infty, +\infty)$. The conjugate exponent of $p > 1$ is denoted by p', that is, $1/p + 1/p' = 1$. The Fourier transform referred to is the complex one; the cosine transform is obtained if $g(-u) = g(u)$. In applying the results to momental means we take $g(u) = g(u) - u$ for $0 \leqq u \leqq 1$ and put $g(u) = 0$, $u > 1$, $g(u) = g(-u)$, $u < 0$. A similar remark, mutatis mutandis, applies in the case of the means of the closed cycle below.

(37)
$$\Omega(h) = \int_{-\infty}^{+\infty} \left| g'(u+h) - g'(u-h) \right| du.$$

On the basis of these and similar criteria it is possible to construct mass functions $q(u)$ which give rise to (F)-effective means. It should be noted that it is neither necessary nor sufficient that $q(u)$ be absolutely continuous for (F)-effectiveness. On the other hand, the reciprocity relations between $g(u)$ and $G(u)$ show that it is necessary that $q(u)$ be equivalent to a continuous function. It follows, in particular, that the (E_p)-means of K. Knopp [24] cannot be (F)-effective since the corresponding function $q(u)$ is a step function. A direct proof of this fact was first given by C. N. Moore [32].

The question of (L)-effectiveness offers additional difficulties. Suppose, however, that (35) holds, and that there exists a positive, monotone decreasing, continuous function $\mathfrak{C}(u)$ such that

(38)
$$\left| C(u) \right| \leqq \mathfrak{C}(u), \quad \int_1^{\infty} \mathfrak{C}(u)du < \infty.$$

Then the corresponding definition $[H, q(u)]$ is (L)-effective, and it will sum the derived series of a function of bounded variation almost everywhere. If a similar inequality is satisfied by the Fourier sine transform of $q(u)-u$, the definition will sum the conjugate series and the conjugate derived series to their proper sums almost everywhere. (A slightly different criterion occurs in [20, third and fourth notes]. See also S. Verblunsky [50].) A definition with all these properties is obtained, in particular, if $Q(u)$ is absolutely continuous and (suitably extended outside of the interval $(0, 1)$) satisfies (37).

9. *Means Based upon two Sequences.* T. H. Gronwall [14] has recently considered means (f, g) based upon two analytic functions $f(w)$ and $g(w)$. Here $f(w)$ is holomorphic for $|w| \leqq 1$, $w \neq 1$, and $z = f(w)$ gives a one-to-one map of $|w| < 1$ on a domain D in $|z| < 1$ so that $z = 0$ or 1 for $w = 0$ or 1 respectively. The inverse function is supposed to be holomorphic on the boundary of D except at $z = 1$ where

(39) $1 - w = (1-z)^{\lambda}\{a + (1-z)\mathfrak{P}(1-z)\}$, $\lambda \geqq 1$, $a > 0$.

Further

$$(40) \qquad g(w) = \sum_{n=0}^{\infty} b_n w^n, \; b_n \neq 0 \text{ for every } n,$$

$$(41) \qquad g(w) = (1 - w)^{-\alpha} + \gamma(w), \; \alpha > 0,$$

where $\gamma(w)$ is holomorphic for $|w| \leqq 1$, and $g(w) \neq 0$ for $|w| < 1$.

Gronwall then associates with the series $\sum_0^\infty u_\nu$ the sequence $\{U_n\}$ defined by the formal identity

$$(42) \qquad \sum_{\nu=0}^{\infty} u_\nu z^\nu = [g(w)]^{-1} \sum_{n=0}^{\infty} b_n U_n w^n,$$

and says that *the series is summable* (f, g) *to the sum* s *if*

$$(43) \qquad \lim_{n \to \infty} U_n = s.$$

All definitions (f, g) corresponding to a $\lambda > 1$ are (L)-effective (with corresponding properties with respect to the associated series) since every series summable (C, α) is also summable (f, g) to the same sum if $\lambda > 1$. The restriction $\lambda > 1$ is essential since, e.g., the (E_p)-means of K. Knopp, already mentioned, are (f, g)-means with $\lambda = 1$, and are known not to be (F)-effective. The particular choice

$$(44) \qquad w = 4z(1 + z)^{-2}, \; g(w) = (1 - w)^{-1/2}$$

with $\lambda = 2$ gives the means of de la Vallée Poussin [49] who proved that this definition is (F)-effective. Generalizations of the means of de la Vallée Poussin are also included among the (f, g)-means.

The (VP)-means are but rarely found as special cases of more general definitions. In addition to Gronwall's (f, g)-means, I have to mention some (F)-effective definitions studied by H. W. Bailey [2] which contain the (VP)-means as special cases.

10. *Means of the Closed Cycle.* I. We return to definitions of summation based on a single generating function which we denote now by $K(s)$. *We suppose* $K(s)$ *to be defined for* $s \geqq 0$ *and to be of bounded variation in* $(0, +\infty)$. *Further,* $K(s)$ *shall be continuous to the right at* $s = 0$ *and* $K(0) = 1$. The $K(s)$-sum of the series $\sum_0^\infty u_n$ is then

$$(45) \qquad \lim_{\omega \to \infty} \sum_{n=0}^{\infty} K\left(\frac{n}{\omega}\right) u_n$$

if the limit exists. The conditions stated are necessary and sufficient for the regularity of this definition. In this paragraph we restrict ourselves to the case in which $K(s) \equiv 0$ for $s \geqq 1$, so that (45) is replaced by

$$\lim_{\omega \to \infty} \sum_{n<\omega} K\left(\frac{n}{\omega}\right) u_n, \tag{46}$$

and the general case is taken up in §11 of Part II.

Special cases of such definitions occur very early in the history of the theory of summability. The discussion of the general case goes back to Fejér [11] and H. Weyl [52]. Fejér proved that such a definition includes $(C, 1)$ if both $s^{2+\rho}|K(s)|$ and $s^{2+\rho}|K''(s)|$ are bounded for $s>1$ and a fixed $\rho>0$. Weyl assumed $K(s)$ to be monotone decreasing to the limit zero. In a different form and applied to Fourier series only, we find a closely related definition in the writings of W. H. Young [55, 56] who works with the formula

$$\tfrac{1}{2}\int_0^\infty [f(x+kt)+f(x-kt)]U_k(t)\,dt = \sum_{n=-\infty}^{+\infty} f_n e^{nix} \int_0^\infty U_k(t) \cos nkt\, dt, \tag{47}$$

where $U_k(t)$ is a function of bounded variation (and absolutely integrable) over $(0, +\infty)$ which vanishes at infinity, and k is supposed to tend to zero. Young limits his actual discussion to the case in which $U_k(t)$ is independent of k, and gives details only for some special cases.

We return to (46) and put

$$C(t) = \int_0^1 K(s) \cos st\, ds. \tag{48}$$

A necessary and sufficient condition for the (F)-effectiveness of this definition of summability is

$$\int_0^\infty |C(t)|\, dt < \infty. \tag{49}$$

The sufficiency is found, in kernel at least, in Young's results. Young of course starts with $C(t)$ and works backwards to $K(s)$.

It is of interest to note that the Fourier transforms play a role already in Weyl's investigation of Gibbs' phenomenon.

Condition (49) is of the same nature as (34) of §8, and what we have said there regarding the construction of (F)-effective kernels applies, mutatis mutandis, to the present situation. Similarly, the condition (38) is sufficient also for the (L)-effectiveness of the $K(s)$-definitions and for the summability almost everywhere of the derived series. The conjugate series involves similar considerations where, however, the cosine transform has to be replaced by the sine transform.

Some special cases require separate mention. In each case it is understood that $K(s) \equiv 0$ for $s > 1$. The choice

$$K(s) = (1 - s^p)^\kappa \tag{50}$$

has been considered by W. H. Young ([55, 56]; details only for $p = 1$ and 2). The case $p = 1$ of course gives the kernel of the Riesz-Cesàro means. The kernel

$$K(s) = (\log C)^\beta (1 - s)^\alpha \left\{ \log \frac{C}{1 - s} \right\}^{-\beta} \tag{51}$$

has been investigated by Bosanquet and Linfoot [4, 5] who have also considered other kernels on the logarithmic scale. They have devoted their attention especially to the relations holding between summability by such kernels and continuity in the mean defined by kernels of the same type. As a further example we may list the kernel

$$K(s) = [\cos \tfrac{1}{2} p\pi s]^\kappa \tag{52}$$

where p and κ are positive integers of which p is odd. This kernel has been introduced by W. Rogosinski [45, 46] in his investigations on section couplings (Abschnittskoppelungen). The corresponding transformation, which is (L)-effective, can be utilized for a number of different purposes in the theory of Fourier series. The case $\kappa = 1$ leads to particularly simple formulas in as much as the generalized sum of the Fourier series is then taken to be

$$\tfrac{1}{2} \lim_{n \to \infty} \left\{ s_n\left(x + \frac{p\pi}{2n}\right) + s_n\left(x - \frac{p\pi}{2n}\right) \right\}. \tag{53}$$

Rogosinski has also investigated other kernels of this general class (even for arbitrary types λ in the sense of M. Riesz), and has proved relative inclusion theorems between his means and those of Riesz.

II. Definitions with Infinite Reference

Of the great number of definitions with infinite reference we can only treat three different classes each of which is of some importance in various branches of analysis. These classes will be the means of the closed cycle, already mentioned in §10, and two types of means used in the theory of analytic continuation.

11. *Means of the Closed Cycle.* II. We return to formula (45). A study of the effectiveness of these definitions with respect to Fourier series has been undertaken by Hille and Tamarkin [unpublished]. If we form the series

$$\sum_{n=-\infty}^{+\infty} K\left(\frac{|n|}{\omega}\right) f_n e^{nix} \tag{54}$$

without assuming that $K(s)$ vanishes for large values of s, it may happen that it does not converge for any values of x and ω for a suitably chosen function $f(x) \subset L$. We can guard ourselves against this by assuming, e.g., that $K(s)$ is monotone decreasing and

$$\int_1^\infty K(s)s^{-1}ds < \infty . \tag{55}$$

This condition ensures the convergence of (54) for $0<\omega<\infty$ whenever $s_n(x) = O(\log n)$, that is, almost everywhere.

But we may also occupy a different point of view (see [22]). We may simply require that whenever $f(x) \subset L$, the series in (54) shall be the Fourier series of a function $f(x, \omega)$ such that

$$f(x, \omega) \to f(x) \tag{56}$$

in some set E_f. A necessary and sufficient condition for (54) to be a Fourier series is that

$$\sum_{n=-\infty}^{+\infty}{}' \frac{1}{n} K\left(\frac{|n|}{\omega}\right) e^{nix} \tag{57}$$

should be the Fourier series of a function of bounded variation (for this range of ideas see M. Riesz [44] where references to the earlier literature are to be found). This condition will be satisfied if we suppose that

$$(58) \qquad \int_0^\infty |C(t)|\,dt < \infty, \quad C(t) = \lim_{a\to+\infty} \int_0^a K(s)\cos st\,ds.$$

Further, if (58) holds, (56) is valid at all points of continuity of $f(x)$, so that the $K(s)$-definition of summation is (F)-effective in this generalized sense, and if (55) is also satisfied it will be (F)-effective in the previous, narrower sense. If there exists a positive, continuous, monotone decreasing function $\mathfrak{E}(t)$ satisfying (38) (of course with $C(t)$ defined by (58)), the definition will also be (L)-effective in the sense that (56) holds in the set where (8) is valid.

The relations between the various means of the closed cycle (Abelian and Tauberian theorems) have recently been elucidated in a brilliant manner by N. Wiener [53]; they are largely governed by the properties of the Fourier transforms of the corresponding kernels. It is likely that the methods of Wiener may be used to great advantage in a study of the effectiveness problems of these definitions. A similar remark would seem to apply to the momental means, the kernels of which appear to be "almost of the closed cycle" in Wiener's terminology.

The so-called "gestrahlte Matrizen" of R. Schmidt [47] define transformations having kernels which usually seem to be "almost of the closed cycle" when not actually "of the closed cycle." The solution of the (F)-effectiveness problem for this class of transformations is likely to be found by Fourier analysis of the kernel.

The means of the closed cycle contain a large number of important special cases. Thus if we choose for $K(s)$ a monotone convex function which tends to zero as $s\to\infty$ and such that

$$(59) \qquad \int_0^1 [1 - K(s)]s^{-1}ds < \infty,$$

we always get an (L)-effective definition (in the narrow sense if (55) or similar condition is satisfied). As classical examples of such a choice we may note the kernels e^{-s} and e^{-s^2} which are

associated with the names Abel, Poisson and Fourier, Poisson, Weierstrass respectively, and which have played an important rôle in the history and development of the theory of Fourier series [1, 12, 38, 51].

The kernel of Riemann [40]

$$s^{-2}\sin^2 s \tag{60}$$

is not of such a simple nature, but is nevertheless (L)-effective. A number of variations and modifications of the Riemann kernel have occurred in the literature [9, 11, 39, 42]. We note especially

$$s^{-p}\sin^p s, \tag{61}$$

$$(-1)^m(2m)!\,s^{-2m}\left\{\cos s - \sum_{k=0}^{m-1}(-1)^k\frac{s^{2k}}{(2k)!}\right\}, \tag{62}$$

$$(-1)^m(2m+1)!\,s^{-2m-1}\left\{\sin s - \sum_{k=0}^{m-1}(-1)^k\frac{s^{2k+1}}{(2k+1)!}\right\}, \tag{63}$$

where m and p are integers $\geqq 1$, which definitions are all (L)-effective. The extreme case $p=1$ in (61) is particularly interesting. This kernel is not of bounded variation in $(0, +\infty)$, and it is known that the corresponding transformation is not regular. Nevertheless, it was shown to be (L)-effective by Lebesgue [27], and we may notice that condition (58) is satisfied.

Definition (45) is a special case of

$$\lim_{\omega\to\infty}\sum_{n=0}^{\infty}K\left(\frac{\lambda_n}{\omega}\right)u_n \quad (0\leqq\lambda_n<\lambda_{n+1},\ \lambda_n\to+\infty) \tag{64}$$

which has been proposed by Perron [37]. In some cases it would seem possible to apply a similar analysis to the (F)-effectiveness problem of such means.

12. *Summation by Analytic Convergence Factors.* The problem of analytic continuation of power series has led to a number of definitions of summation. Among these we may distinguish various definitions employing convergence factors $\gamma(n, \alpha)$ which are holomorphic functions of n as well as of α and α is supposed to tend to zero. One possibility of getting such factors is to put $\gamma(n, \alpha)=K(\alpha, n)$ where $K(z)$ satisfies the conditions of regularity stated in §§10 and 11, and, in addition, is holomorphic in

a domain of the complex z-plane containing the positive real axis including the origin. In addition to the choices e^{-z} and e^{-z^2} already mentioned, we may list the kernels

$$(65)\qquad \frac{1}{\Gamma(1+z)},\quad \frac{\cos \pi z}{\Gamma(1+z)},\quad \frac{\sin \pi z}{\Gamma(1+z)},\quad \frac{e^{+\pi i z}}{\Gamma(1+z)},$$

of which the first one has been used by G. Mittag-Leffler [31] and the others by H. von Koch [25]; all these definitions are (L)-effective.

Definitions of this nature, but based upon a more general choice of $\gamma(z, \alpha)$, have been proposed by E. Lindelöf [30]. These definitions embrace as particular cases those of von Koch and Mittag-Leffler quoted above, as well as the definition of Le Roy [29] and one by Lindelöf himself corresponding to

$$(66)\qquad \frac{\Gamma(1+(1-\alpha)z)}{\Gamma(1+z)},\ \exp\left\{-\alpha(z+1)\log(z+1)\right\}$$

respectively. Both these definitions are (L)-effective since any series summable $(C, 1)$ is summable to the same sum by either method (H. L. Garabedian [13], and D. S. Morse [33], respectively). As a further special case we may mention a definition due to Perron [37], based on the convergence factors

$$(67)\qquad \frac{\omega^{z+1}\Gamma(\omega)}{\Gamma(\omega+z+1)},\ \omega=\alpha^{-1},$$

which is easily shown to be (L)-effective.

The Dirichlet series definitions studied by H. L. Garabedian [13] and W. H. Durfee [10] with

$$(68)\qquad \gamma(n, \alpha)=e^{-\alpha\lambda_n},\ 0\leqq\lambda_n<\lambda_{n+1},\ \lambda_n\to+\infty,$$

are special cases of Lindelöf's general definition if λ_n is a sufficiently simple analytic function of n. The inclusion relations holding between these definitions and the typical means of M. Riesz show that they are ordinarily (L)-effective unless the sequence $\{\lambda_n\}$ grows either too fast or too slowly.

13. *Summation by Entire Functions.* Another type of means which has been utilized in order to effect analytical continuation of power series is based upon the use of summatory functions,

especially entire functions. We take as the generalized limit of the sequence $\{s_n\}$ the expression

$$\lim_{\omega\to\infty} [E(\omega)]^{-1} \sum_{n=0}^{\infty} s_n c_{n+1}\omega^{n+1}, \tag{69}$$

where

$$E(z) = \sum_{n=0}^{\infty} c_n z^n,\ c_0 = 1,\ c_n \geqq 0, \tag{70}$$

is an entire function. Such definitions have been proposed by Borel, Lindelöf, Mittag-Leffler, and others (see A. Buhl [6]). It seems that these definitions are ordinarily not even (F)-effective. This was shown by C. N. Moore [32] in the case of Borel's own definition corresponding to $E(z)=e^z$. It is also true of the more powerful definitions based on Mittag-Leffler's function $E_\alpha(z)$, and rough estimates indicate that the same negative result is valid for a wide class of definitions such that the coefficients c_n decrease to zero in a sufficiently regular manner [Hille and Tamarkin, unpublished]. The two classes of definitions in §§12 and 13 have about the same capacity of handling power series, but their effectiveness with respect to Fourier series appears to be entirely different.

III. Definitions for Summation of Integrals

14. *Summable Integrals.* The nth partial sum of a Fourier series is given by formula (10), i.e. by a definite integral involving a parameter n. This suggests a possible application of the theory of summable integrals to our problem. I shall restrict myself to one single illustration, viz. to a definition due to F. Nevanlinna [35] which has close relations to the means of the closed cycle. Let $K(u)>0$ and

$$\int_0^1 K(u)du = 1. \tag{71}$$

Putting

$$F_K(\omega) = \int_0^1 K(u)F(\omega u)du, \tag{72}$$

we can affirm that

$$\lim_{\omega\to\infty} F_K(\omega) = \lim_{\omega\to\infty} F(\omega), \tag{73}$$

whenever the right-hand side exists and is finite. F. Nevanlinna replaces the integral of Dirichlet by the integral of Fourier over a finite interval

$$F(\omega; x) = \frac{1}{\pi}\int_a^b f(t)\frac{\sin\omega(t-x)}{t-x}\,dt, \tag{74}$$

which has similar convergence and summability properties. He takes this as the function $F(\omega)$ and supposes, in addition, that $K(u)$ is monotone increasing in $0 \leqq u \leqq 1$, and that

$$\lim_{\epsilon\to 0}\int_0^{1-\epsilon} K(u)\log\frac{1}{1-u}\,du < \infty. \tag{75}$$

He then shows that

$$\frac{1}{\pi}\int_0^1 K(u)\int_a^b f(t)\frac{\sin\omega u(t-x)}{t-x}\,dt\,du \to f(x) \tag{76}$$

at every point of continuity. It follows that these definitions of summability are (F)-effective. A. F. Moursund [34] has recently shown that they are also (L)-effective.

15. *Conclusion.* The preceding survey is rather incomplete. In restricting the attention to the (F)- and (L)-effectiveness problems, I had to disregard a number of important developments. The reader will undoubtedly miss references to the theory of multiple series, conjugate series and derived series, to various types of summability, to convergence in the mean, strong convergence and various other forms of generalized convergence, to Tauberian theorems, and to a number of other topics. In spite of having obviously given too little, I may also with due cause be taken to task for having given too much. But I shall be happy if I have at least convinced the reader that the regular definitions of summability which are effective with respect to Fourier series form a vast and important class, and that there is some hope of our being able to bring some order into the bewildering chaos which reigns in this domain at present.

16. *Bibliography.* The subsequent list of references contains only papers referred to in this report, and no pretence is made of its being in any way complete.

Bibliography

1. N. H. Abel, *Untersuchungen über die Reihe . . .*, Journal für Mathematik, vol. 1 (1826), pp. 311–338.

2. H. W. Bailey, *The summability of single and multiple Fourier series*, Annals of Mathematics, (2), vol. 28 (1926–7), pp. 69–91.

3. E. Borel, *Mémoire sur les séries divergentes*, Annales de l'Ecole Normale, (3), vol. 16 (1899), pp. 9–131, pp. 132–136.

4. L. S. Bosanquet and E. H. Linfoot, *Generalized means and the summation of Fourier series*, Quarterly Journal of Mathematics (Oxford), vol. 2 (1931), pp. 207–229.

5. L. S. Bosanquet and E. H. Linfoot, *On the zero order summability of Fourier series*, Journal of the London Mathematical Society, vol. 6 (1931), pp. 117–126.

6. A. Buhl, *Séries analytiques. Sommabilité*, Mémorial des Sciences Mathématiques, No. 7 (1925), 53 pp.

7. S. Chapman, *On the general theory of summability, with application to Fourier's and other series*, Quarterly Journal, vol. 43 (1911), pp. 1–52.

8. S. Chapman, *On non-integral orders of summability of series and integrals*, Proceedings of the London Mathematical Society, (2), vol. 9 (1911), pp. 369–409.

9. N. Cioranescu, *Sur la sommation des séries trigonométriques*, Atti dei Lincei, Rendiconti, (6), vol. 13 (1931), pp. 577–582. See the comments in a review by E. Hille, Zentralblatt, vol. 2 (1931), pp. 130.

10. W. H. Durfee, *Summation factors which are powers of a complex variable*, American Journal of Mathematics, vol. 53 (1931), pp. 817–842.

11. L. Fejér, *Untersuchungen über Fouriersche Reihen*, Mathematische Annalen, vol. 58 (1903), pp. 51–69.

12. J. J. Fourier, *Théorie Analytique de la Chaleur*, Paris, 1822, =Œuvres, vol. I (Chap. IV, §279).

13. H. L. Garabedian, *On the relation between certain methods of summability*, Annals of Mathematics, (2), vol. 32 (1931), pp. 83–106.

14. T. H. Gronwall, *Summation of series and conformal mapping*, Annals of Mathematics, (2), vol. 33 (1932), pp. 101–117.

15. G. H. Hardy, *Theorems relating to the summability and convergence of slowly oscillating series*, Proceedings of the London Mathematical Society, (2), vol. 8 (1910), pp. 301–320.

16. G. H. Hardy, *On the summability of Fourier series*, Proceedings of the London Mathematical Society, (2), vol. 12 (1913), pp. 365–372.

17. G. H. Hardy, *The summability of a Fourier series by logarithmic means*, Quarterly Journal of Mathematics (Oxford), vol. 2 (1931), pp. 107–112.

18. G. H. Hardy and M. Riesz, *The general theory of Dirichlet's series*, Cambridge Mathematical Tracts, No. 18 (1915), 78 pp.

19. F. Hausdorff, *Summationsmethoden und Momentfolgen*, I, II, Mathematische Zeitschrift, vol. 9 (1921), pp. 74–109, 280–299.

20. E. Hille and J. D. Tamarkin, *On the summability of Fourier series*, four notes, Proceedings National Academy of Sciences, vol. 14 (1928), pp. 915–918; vol. 15 (1929), pp. 41–42; vol. 16 (1930), pp. 594–598; vol. 17 (1931), pp. 376–380.

21. E. Hille and J. D. Tamarkin, *On the summability of Fourier series*, I, to appear in Transactions of this Society, vol. 34 (1932).

22. E. Hille and J. D. Tamarkin, *On the summability of Fourier series*, II, to appear in Annals of Mathematics.

23. W. A. Hurwitz and L. L. Silverman, *On the consistency and equivalence of certain definitions of summability*, Transactions of this Society, vol. 18 (1917), pp. 1–20.

24. K. Knopp, *Über das Eulersche Summierungsverfahren*, I, II, Mathematische Zeitschrift, vol. 15 (1922), pp. 226–253; vol. 18 (1923), pp. 125–156.

25. H. von Koch, *Contributions à la théorie des prolongements d'une fonction analytique*, Arkiv för Matematik, vol. 12, no. 23 (1917), 48 pp.

26. H. Lebesgue, *Recherches sur la convergence des séries de Fourier*, Mathematische Annalen, vol. 61 (1905), pp. 251–280.

27. H. Lebesgue, *Leçons sur les Séries Trigonométriques*, Paris, 1906, 128 pp.

28. H. Lebesgue, *Sur les intégrales singulières*, Annales de Toulouse, (3), vol. 1 (1909), pp. 25–117.

29. E. LeRoy, *Sur les séries divergentes et les fonctions définies par un développement de Taylor*, Annales de Toulouse, (2), vol. 2 (1900), pp. 317–430.

30. E. Lindelöf, *Sur l'application de la théorie des résidus au prolongement analytique des séries de Taylor*, Journal de Mathématiques, (5), vol. 9 (1903), pp. 213–221.

31. G. Mittag-Leffler, *Sur la représentation arithmétique des fonctions analytiques générales d'une variable complexe*, Atti del IV Congresso, Roma, 1908, vol. I (1909), pp. 67–85.

32. C. N. Moore, *On the application of Borel's method to the summation of Fourier's series*, Proceedings of the National Academy Sciences, vol. 11 (1925), pp. 284–287.

33. D. S. Morse, *Certain definitions of summability*, American Journal of Mathematics, vol. 45 (1923), pp. 259–285.

34. A. F. Moursund, *On a method of summation of Fourier series*, Annals of Mathematics, (2), vol. 33 (1932).

35. F. Nevanlinna, *Über die Summation der Fourier'schen Reihen und Integrale*, Översikt av Finska Vetenskaps-Societetens Förhandlungen, vol. 64(A), no. 3 (1920–1), 14 pp.

36. N. E. Nörlund, *Sur une application des fonctions permutables*, Lunds Universitets Arsskrift, N.F., (2), vol. 16, no. 3 (1920), 10 pp.

37. O. Perron, *Beitrag zur Theorie der divergenten Reihen*, Mathematische Zeitschrift, vol. 6 (1920), pp. 286–310.

38. S. D. Poisson, *Mémoire sur la distribution de la chaleur dans les corps solides*, Journal de l'Ecole Polytéchnique, vol. 19 (1823), pp. 1–144.

39. A. Rajchman and A. Zygmund, *Sur la relation du procédé de sommation de Cesàro et celui de Riemann*, Bull. Int. Acad. Polon. Cl. sci. math. et nat. A, 1925, pp. 69–80.

40. B. Riemann, *Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe*, Göttinger Abhandlungen, vol. 13 (1867) = Werke, vol. 1, pp. 227–271.

41. M. Riesz, *Sur les séries de Dirichlet et les séries entières*, Comptes Rendus, vol. 149 (22 Nov. 1909), pp. 909–912.

42. M. Riesz, *Über summierbare trigonometrischen Reihen*, Mathematische Annalen, vol. 71 (1912), pp. 54–75.

43. M. Riesz, *Sur l'équivalence de certaines méthodes de sommation*, Proceedings of the London Mathematical Society, (2), vol. 22 (1924), pp. 412–419.

44. M. Riesz, *Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires*, Acta Mathematica, vol. 49 (1927), pp. 465–497.

45. W. Rogosinski, *Über die Abschnitte trigonometrischer Reihen*, Mathematische Annalen, vol. 95 (1925), pp. 110–134.

46. W. Rogosinski, *Reihensummierung durch Abschnittskoppelungen*, I, Mathematische Zeitschrift, vol. 25 (1926), pp. 132–149.

47. R. Schmidt, *Über divergente Folgen und lineare Mittelbildungen*, Mathematische Zeitschrift, vol. 22 (1925), pp. 89–152.

48. M. H. Stone, *The summability of Fourier series*, this Bulletin, vol. 33 (1927), pp. 721–732.

49. Ch.-J. de la Vallée Poussin, *Sur l'approximation des fonctions d'une variable réelle et de leurs dérivées par des polynomes et de suites limitées de Fourier*, Académie Belgique, Bulletin Classe scientifique, 1908, pp. 193–254.

50. S. Verblunsky, *The convergence of singular integrals*, Proceedings Cambridge Philosophical Society, vol. 26 (1929–30), pp. 312–322.

51. K. Weierstrass, *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen*, Berliner Sitzungsberichte, 1885, pp. 633–639, pp. 789–805 = Werke, vol. 3, pp. 1–37.

52. H. Weyl, *Über die Gibbs'sche Erscheinung und verwandte Konvergenzphänomene*, Rendiconti di Palermo, vol. 30 (1910), pp. 377–407.

53. N. Wiener, *Tauberian theorems*, Annals of Mathematics, (2), vol. 33 (1932), pp. 1–100.

54. G. F. Woronoi, *Extension of the notion of the limit of the sum of an infinite series* (in Russian), Proceedings XI Congress (1901) of Russian Nat. and Physicians, St. Petersburg, 1902, pp. 60–61. Annoted English translation by J. D. Tamarkin, in Annals of Mathematics, (2), vol. 33 (1932) pp. 422–428.

55. W. H. Young, *Über eine Summationsmethode für die Fouriersche Reihe*, Leipziger Berichte, vol. 36 (1911), pp. 369–387.

56. W. H. Young, *On infinite integrals involving a generalisation of the sine and cosine functions*, Quarterly Journal of Mathematics, vol. 43 (1912), pp. 161–177.

PRINCETON UNIVERSITY

NOTE ON THE LAW OF BIQUADRATIC RECIPROCITY*

BY V. W. ADKISSON

In outlining the proof of the law of biquadratic reciprocity H. J. S. Smith develops the expressions for S, T, S^4, and T^4† which are used in the proof given by Eisenstein.‡ We give here a slightly different development for these values making use of certain relationships established by Lebesgue.§ The advantage in this development lies in the fact that the function $\psi(i)$ is exhibited in a form which shows it to be a polynomial in i with integral coefficients, and that $\pm\psi(i)$ is a primary prime in the realm $k(i)$ if the proper sign be chosen. If

$$F(\alpha) = \sum_{n=0}^{p-2} \alpha^n x^{g^n},$$

where α is a root of the equation $(\alpha^{p-1}-1)/(\alpha-1)=0$, x is a root of $(x^p-1)/(x-1)=0$, g is a primitive root of p, and p is a prime of the form $4n+1$, then

$$F(\alpha)F(\alpha^{-1}) = \alpha^{(p-1)/2}p. \tag{1}$$

Substituting i for α, we obtain the result

$$F(i)F(i) = F(-1)\sum_{t=1}^{p-2} i^{\text{ind}\,t}(-1)^{\text{ind}\,(t+1)}.\| \tag{2}$$

Let

$$\psi(i) = \frac{[F(i)]^2}{F(-1)} = \sum_{t=1}^{p-2} i^{\text{ind}\,t}(-1)^{\text{ind}\,(t+1)}. \tag{3}$$

Hence, $\psi(i)$ is a polynomial in i with integral coefficients, and may be written in the form $a+bi$ where a and b are integers. But

---

* Research paper No. 280, University of Arkansas.

† H. J. S. Smith, *Collected Mathematical Papers*, vol. 1, pp. 78–87.

‡ E. Eisenstein, *Lois de réciprocité*, Journal für Mathematik (Crelle), vol. 28, pp. 57–67.

§ L. M. V.-A. Lebesgue, *Démonstration de quelques formules d'un mémoire de M. Jacobi*, Journal de Mathématiques (Liouville), vol. 19, pp. 289.

‖ Lebesgue, loc. cit.

$$(4) \qquad \psi(i)\psi(-i) = \frac{[F(i)F(-i)]^2}{[F(-1)]^2} = \frac{[i^{(p-1)/2}p]^2}{(-1)^{(p-1)/2}p} = p.$$

Therefore, $\psi(i)$ and $\psi(-i)$ are primes in $k(i)$. Since there are an even number of terms in (3) for which $\operatorname{ind} t$ is odd, and an odd number of terms for which $\operatorname{ind} t$ is even, $\pm\psi(i)$ and $\pm\psi(-i)$ will be primary primes, $a+bi$ and $a-bi$ respectively, in $k(i)$ if the proper signs be taken in each case. Moreover, $\psi(-i)$ [or $-\omega(-i)$] is primary if $\psi(i)$ [or $-\psi(i)$] is primary, and hence $(a+bi)(a-bi)=p$.

Since g is a primitive root of p, we have

$$g^{(p-1)/2} \equiv -1, \bmod p; \quad g^{(p-1)/4} \equiv i, \quad \bmod p_1;$$

and

$$g^{(p-1)/4} \equiv -i, \bmod p_2,$$

where $p_1p_2=p$. Then $(g/p_1)_4=i$.*

Let $g^s \equiv k \bmod p$; then since

$$F(i) = \sum_{s=0}^{p-2} i^s x^{g^s},$$

we find

$$F(i) = \sum_{s=0}^{p-2} \left(\frac{g^s}{p_1}\right)_4 x^{g^s} = \sum_{k=1}^{p-1} \left(\frac{k}{p_1}\right)_4 x^k = S.$$

From (3) and (4),

$$[F(i)]^4 = [F(-1)]^2[\psi(i)]^2 = p(a+bi)^2 = pp_1^2 = S^4.$$

In like manner

$$F(-i) = \sum_{k=1}^{p-1} \left(\frac{k}{p_1}\right)_4^3 x^k = T,$$

and

$$T^4 = p(a-bi)^2 = pp_2^2.$$

UNIVERSITY OF ARKANSAS

* The symbol $(g/p_1)_4$ due to H. J. S. Smith is the power of i which is congruent to $g^{(p-1)/4} \bmod p_1$.

A PLANE, ARCWISE CONNECTED AND CONNECTED IM KLEINEN POINT SET WHICH IS NOT STRONGLY CONNECTED IM KLEINEN*

BY R. L. WILDER

In a recent paper,† G. T. Whyburn gave an example, in three-dimensional space, of an arcwise connected and connected im kleinen point set which is not arcwise connected im kleinen, and raised the question as to whether such a set can exist in the plane; also, whether there exists in the plane a strongly connected and connected im kleinen set which is not strongly connected im kleinen. The purpose of this note is to answer both of these questions affirmatively. Indeed, there exists in the plane a point set M which is arcwise connected and connected im kleinen, but which fails *at every point* to be strongly connected im kleinen.

Let S denote the set of all points of the plane that lie within or on the square whose vertices are the points (0,0), (1,0), (1,1), (0,1); denote these vertices by A, B, C, D, respectively. The open interval $<AB>$ is the sum of c (where c is the cardinal number of the continuum) mutually exclusive sets each of which is dense in AB.‡ Denote the class of these sets by X.

Let K denote the class of all continua contained in S, which contain at least one point of each of the straight line intervals AD and BC, but no point of AB or CD. This class has the cardinal number c.

There exists a one-to-one correspondence, T, between the elements of the classes X and K. Let $E(X)$ and $E(K)$ be elements of X and K, respectively, which correspond to one another under T. Then, if x is a point of $E(X)$, denote by l_x the set of points of S that lie on the vertical line through x, and let P_x denote the

---

* Presented to the Society, December 31, 1930.

† *Concerning points of continuous curves defined by certain im kleinen properties*, Mathematische Annalen, vol. 102 (1929), pp. 313–336.

‡ See Knaster and Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255; see especially p. 252. This paper will be referred to hereafter as S.E.C.

first point (in ascending order of ordinates) of $E(K)$ on l_x. The set of all points P_x obtained from $E(K)$ in this way denote by $N_{E(X)}$. Then let $N=\sum_{E(X)} N_{E(X)}$, and let $M=S-N$.

The set N is punctiform.* Also, the set M is obviously arcwise connected; for if P and Q are points of M, there is a broken line of M connecting P and Q, consisting of portions of the broken line $ABCDA$ and of the vertical lines through P and Q. Furthermore, M is connected im kleinen. Suppose, for instance, P is a point of M lying within the square $ABCD$. Let $S(P)$ denote a circle with center at P and lying wholly within S, and let R denote the set of all points of M that lie within $S(P)$. Then R is connected. For suppose R is the sum of two mutually separated sets R_1 and R_2. Then there exists† a continuum F which separates the plane between two points P_1 and P_2 belonging to R_1 and R_2, respectively, and which contains no point of R. As F has points within $S(P)$, there is a subcontinuum F' of F which lies wholly within $S(P)$ and is therefore a subset of N. As N is punctiform, this is of course impossible. Consequently R is connected, and M is connected im kleinen.

It remains only to observe that M is not strongly connected im kleinen at any point. Suppose, for instance, P is a point of M within $ABCD$. Define $S(P)$ as above, and let $S'(P)$ be a circle concentric with and lying within $S(P)$. Let Q be a point of M interior to $S'(P)$, but not on the same vertical as P. A continuum H joining P and Q and interior to $S(P)$ belongs to some $E(K)$ of such a character that H contains points of N. Consequently M is not strongly connected im kleinen at P. If P lies on $ABCDA$, say on AB, we choose Q so that the point of N on the vertical through Q lies below Q, and employ a suitably chosen subcontinuum H' of H that does not meet $ABCDA$.

THE UNIVERSITY OF MICHIGAN

* See S.E.C., pp. 245–246.

† See S.E.C., Theorem 37.

SOME INVOLUTORIAL LINE TRANSFORMATIONS INTERPRETED AS POINTS OF V_2 OF S_5*

BY J. M. CLARKSON

1. *Introduction.* Consider the identity

$$\text{I} \qquad V_2(x) \equiv x_1x_4 + x_2x_5 + x_3x_6 \equiv 0$$

existing among the Plücker coordinates $x_1, x_2, \cdots, x_6$ of a line in S_3 as the equation of a quadratic hypersurface in S_5. The existence of a (1,1) correspondence between the lines of S_3 and the points of V_2 is well known, as is also the representation of ruled surfaces in S_3 by curves of the same orders and genera on V_2.†

To a bundle of lines in S_3 corresponds a plane of points on V_2, and to a plane field of lines corresponds a plane of points on V_2, called ω-planes and ρ-planes, respectively, throughout this paper. The two systems of planes are each ∞^3, and there are certain relations among them. Two planes of either system have always one and only one point in common, corresponding to the line common to the two representative bundles or plane fields; and a plane of the ω-system meets a plane of the ρ-system either in no point or in a line of points, corresponding to the flat pencil common to a bundle and a plane field of S_3 when the vertex of the bundle lies on the plane of the field.

Line transformations of S_3 are, therefore, point transformations on V_2. When the point transformations on V_2 are non-linear, their fundamental elements may be of dimension 0, 1, 2, 3, and their images, or the principal elements in the transformations, may be of dimension 1, 2, 3, 4. The transformations considered below are birational, but since the equation of V_2 does not enter into the discussion of their birationality, we conclude that they are Cremona transformations for all of S_5.

2. *Three Involutorial Transformations on V_2.*

CASE 1. J. DeVries‡ discusses synthetically several involutions

* Presented to the Society, March 26, 1932.

† See, for example, W. L. Edge, *Ruled Surfaces.*

‡ *Proceedings, Koninklijke Akademie van Wetenschappen te Amsterdam*, vol. 22 (1920), pp. 478–481, 634–640.

among the rays of space. It shall be our purpose here to obtain analytic formulations of these transformations on V_2 in S_5. To this end, let us consider the four flat pencils (A_i, α_i) in the planes α_i and with vertices at A_i. Consider the planes α_i as being the faces of the tetrahedron of reference so that $\alpha_i \equiv z_i = 0$, where z_i are the variable coordinates of a point of S_3, and choose for the points A_1, A_2, A_3, A_4 the points (0,1,1,1), (1,0,1,1), (1,1,0,1), (1,1,1,0) respectively. The pencils (A_i, α_i) are therefore independent and the Plücker coordinates of a line of each are

$$(1) \qquad \begin{cases} a_1 \equiv [(\lambda_1 + \lambda_2), -\lambda_1, -\lambda_2, 0, 0, 0], \\ a_2 \equiv [-(\mu_1 + \mu_2), 0, 0, 0, \mu_2, -\mu_1], \\ a_3 \equiv [0, -(l_1 + l_2), 0, -l_2, 0, l_1], \\ a_4 \equiv [0, 0, -(m_1 + m_2), m_2, -m_1, 0], \end{cases}$$

where the λ_j, μ_j, l_j, m_j, represent parameters and the a_i lines from the pencils (A_i, α_i) respectively.

An arbitrary line (y) will determine a definite line of each pencil, and these four lines, in general mutually skew, will have another transversal line (x). Conversely, the line (x) determines the line (y), and hence if we consider the variables (x_i), (y_i) as coordinates of points in S_5, we have an involution of S_5 which leaves V_2 invariant. Such a line (y) determines in the given four pencils the lines

$$(2) \qquad \begin{cases} a_1 \equiv [(y_5 - y_6), (y_6 - y_4), (y_4 - y_5), 0, 0, 0], \\ a_2 \equiv [-(y_2 + y_3), 0, 0, 0, (y_3 + y_4), (y_4 - y_2)], \\ a_3 \equiv [0, (y_1 + y_3), 0, (y_3 - y_5), 0, (y_1 + y_5)], \\ a_4 \equiv [0, 0, -(y_1 + y_2), (y_2 + y_6), (y_6 - y_1), 0]. \end{cases}$$

Recalling that the lines meeting a given line a of S_3 are represented by the intersection with V_2 of the S_4 tangent to it at the point which represents a, we have the equations

$$(3) \qquad \begin{cases} (y_5 - y_6)x_4 + (y_6 - y_4)x_5 + (y_4 - y_5)x_6 = 0, \\ (y_3 + y_4)x_2 + (y_4 - y_2)x_3 - (y_2 + y_3)x_4 = 0, \\ (y_3 - y_5)x_1 + (y_1 + y_5)x_3 + (y_1 + y_3)x_5 = 0, \\ (y_2 + y_6)x_1 + (y_6 - y_1)x_2 - (y_1 + y_2)x_6 = 0, \end{cases}$$

of four S_4's tangent to V_2 at the points a_1, a_2, a_3, a_4 respectively.

The problem of solving four linear and one quadratic equations in six unknowns is the same as that discussed by V. Snyder,* in which he determines the two lines common to four linear line-complexes. Using his method of solution we find

$$(4)\qquad \begin{aligned} \rho x_1 &= w_1 Q\cdot R, \quad \rho x_2 = w_2 S\cdot R, \quad \rho x_3 = w_3 S\cdot Q,\\ \rho x_4 &= w_4 S\cdot U, \quad \rho x_5 = w_5 U\cdot Q, \quad \rho x_6 = w_6 U\cdot R, \end{aligned}$$

where the w_i are linear and Q, R, S, U are cubic in y_j. Thus the transformation (4) is an involution of order seven.

Since V_2 is invariant under (4), x_i must satisfy $V_2(x)=0$. But if this be true, $V_2(w)=0$ also.

If $Q(y)=0$, then in (4) $x_1=x_3=x_5=0$. Hence any point in the plane $x_1=x_3=x_5=0$ on V_2 is transformed into the intersection of the cubic hypersurface $Q(x)=0$ and $V_2(x)=0$. Likewise any point in the plane $x_1=x_2=x_6=0$ is transformed into the intersection of $R(x)=0$ and $V_2(x)=0$, any point in the plane $x_2=x_3=x_4=0$ is transformed into the intersection of $S(x)=0$ and $V_2(x)=0$, and any point in the plane $x_4=x_5=x_6=0$ is transformed into the intersection of $U(x)=0$ and $V_2(x)=0$. These planes on V_2 are therefore fundamental elements of (4). They are ρ-planes, being the representation on V_2 of the lines of the planes α_1, α_2, α_3, α_4 respectively. Dually, the ω-planes, representative of the bundles of lines of S_3 with vertices at A_1, A_2, A_3, A_4 respectively, are fundamental elements of (4).

There are ρ-planes and ω-planes which carry singular flat pencils. Consider the ρ-plane which represents the lines of the plane $A_1A_2A_3$ in S_3. Its equations are

$$(4a_1)\qquad x_4+x_5-2x_6=0;\ 2x_2-x_3+x_4=0;\ 2x_1+x_3+x_5=0;$$

and the point (0,0,1,1,−1,0) which lies in this plane is such that every line in the pencil with this point as vertex and ($4a_1$) as base is singular. In S_3, plane $A_1A_2A_3$ meets α_4 in a line. Every line t in $A_1A_2A_3$ meets the intersection $A_1A_2A_3$, α_4 in a point and so determines a line a_4. The line t meets each of the intersections $A_1A_2A_3$, α_1; $A_1A_2A_3$, α_2; $A_1A_2A_3$, α_3. These intersections are lines of the pencils (A_i, α_i), $(i=1, 2, 3)$ respectively, and t', the conjugate of t in (4), is any one of the lines of the pencil in the plane

* This Bulletin, vol. 3 (1897), pp. 247–250.

$A_1A_2A_3$ with vertex t, a_4. Hence, the pencil is singular. But, on V_2 in S_5, a pencil of lines of S_3 is represented by a line and since all of the pencils $(t, a_4; A_1A_2A_3)$ have in common the line $A_1A_2A_3$, α_4 whose representation on V_2 is the point (0,0,1,1,−1,0), the lines on V_2 which represent the pencils $(t, a_4; A_1A_2A_3)$ intersect in this point. In like manner, each of the planes

$$(4a_2)\quad x_4 - 2x_5 + x_6 = 0;\ 2x_1 + x_2 - x_6 = 0;\ x_2 + 2x_3 + x_4 = 0;$$

$$(4a_3)\quad 2x_4 - x_5 - x_6 = 0;\ x_1 + 2x_3 - x_5 = 0;\ x_1 + 2x_2 + x_6 = 0;$$

$$(4a_4)\quad x_2 - x_3 - 2x_4 = 0;\ x_1 - x_3 + 2x_5 = 0;\ x_1 - x_2 - 2x_6 = 0;$$

carries a singular pencil of lines with vertices at the points (0,1,0,−1,0,1), (1,0,0,0,1,−1), (1,1,1,0,0,0) respectively. Dually, each of the ω-planes on V_2 which represent the bundles of lines of S_3 with vertices at $\alpha_1\alpha_2\alpha_3$, $\alpha_1\alpha_2\alpha_4$, $\alpha_1\alpha_3\alpha_4$, $\alpha_2\alpha_3\alpha_4$ carries a singular pencil of lines.

The involution (4) has fundamental points which are transformed into quadric surfaces lying on V_2. Consider the point (1,0,0,0,0,0). If $U=S=0$, then application of (4) gives this point. But, if

$$(5a_1)\qquad x_3 - x_5 = 0;\ x_2 + x_6 = 0,$$

then

$$U(x) = S(x) = 0.$$

But $(5a_1)$ are the equations of a three-space common to the four-spaces $x_1=0$, $x_4=0$, both of which are tangent to V_2. Thus $(5a_1)$ are the equations of a quadric surface on V_2. In like manner we find that the points

$$(0,1,0,0,0,0),(0,0,1,0,0,0),(0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)$$

are fundamental points with quadric surfaces as images. Dually, the points

$$(0,1,-1,1,1,1),(-1,0,1,1,1,1),(1,-1,0,1,1,1),$$
$$(1,1,-1,1,-1,0),(1,-1,1,-1,0,1),(-1,1,1,0,1,-1)$$

are fundamental with quadrics as images.

The locus of invariant points of (4) is the intersection of V_2 and a hypersurface of order four. Snyder* showed that the two lines common to four linear complexes are coincident if, and only if, the combinant

* Loc. cit.

$$(6) \qquad \Delta \equiv \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix}$$

of the system of complexes having these four as base vanishes. Equations (3) are the equations of four special linear complexes, and the combinant of the system with (3) as base is Δ, where

$$A_{ik} \equiv a_{i1}a_{k4} + a_{i2}a_{k5} + a_{i3}a_{k6} + a_{i4}a_{k1} + a_{i5}a_{k2} + a_{i6}a_{k3},$$

where now a_{ij} represents the coefficient of x_j in the ith equation of (3).

After making the substitution indicated above and using the fact that $V_2(w) = 0$, we obtain

$$(7) \qquad \begin{aligned} \Delta(x) &\equiv [w_2(x)w_5(x)x_3x_6 - w_3(x)w_6(x)x_2x_5]^2 \\ &\equiv [w_1(x)w_4(x)x_3x_6 - w_3(x)w_6(x)x_1x_4]^2 \\ &\equiv [w_1(x)w_4(x)x_2x_5 - w_2(x)w_5(x)x_1x_4]^2. \end{aligned}$$

Thus, the locus of invariant points of (4) is

$$(8) \qquad w_1w_4x_2x_5 - w_2w_5x_1x_4 = 0; \; V_2(x) = 0.$$

Among the lines of S_3, (4) transforms a pencil (T, τ) into a ruled surface R_7 of order seven. Four lines of the pencil (T, τ) are generators of R_7, for (8) is the equation of a line-complex of order four, of which *any* pencil carries four lines. On R_7 there is a double curve of order 15 and genus 6.* The plane τ intersects R_7 in four lines t_1, t_2, t_3, t_4 of (T, τ) and a curve of order 3 which must pass through T since every generator of R_7 meets five others. Hence there are two generators of R_7 not in τ met by each of the lines t_i. This cubic curve is rational, since the surface R_7 is rational, and has a double point which is not on any of the lines t_i. The double curve of order 15 meets τ in 15 points, of which 8 are the pairs of points in which t_i meet the residual cubic $[R_7, \tau]$. The six pairs of generators of R_7 among t_i show that the double curve has a 6-fold point at T. Project this curve from T upon a plane. We obtain thus a plane curve ρ_9 of

* See W. L. Edge, *Ruled Surfaces*, pp. 9 and 29.

order 9 and genus 6. The line of intersection of τ and the plane of section passes through four double points of ρ_9 and hence meets ρ_9 in one other point, the point where the line of τ through T and the double point of $[R_7, \tau]$ meets the plane of section. Now, R_7 has 10 triple points,* of which the four-fold point T counts for four. Hence there are six others which are also three-fold on the double curve. These six triple points project into triple points of ρ_9, and since ρ_9 is of genus 6 and has four double points (mentioned above), these six triple points are just sufficient to complete the singularities of ρ_9.

On V_2 in S_5 there are six ω-planes with three intersections each with C_7 (representation of R_7). These are the ω-planes which represent the bundles of lines through the triple points of R_7. Likewise, there are six ρ-planes trisecant to C_7, representing the lines of the tritangent planes of R_7. There is one plane of each system having four intersections with C_7, representing the bundle T and the plane field τ respectively. There are only ∞^1 planes of V_2 which meet C_7 twice, and ∞^2 which meet C_7 once. They represent the lines of the bundles and of the bitangent planes at points of the double curve, and the lines of the bundles and of the tangent planes at the ordinary points of R_7 respectively.

A bundle of lines T is transformed into a line-congruence of order seven formed by the generators of the ruled surfaces R_7 belonging to the singly infinite system arising from the pencils of lines through T. The invariant lines of the transformation make up a quartic cone K_4 which must pass through the points A_1, A_2, A_3, A_4; $\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_4$; $\alpha_1, \alpha_3, \alpha_4, \alpha_2, \alpha_3, \alpha_4$, since each line t of the bundles having these points as vertices is transformed into either a quadric regulus to which t belongs or into a pencil of lines containing t.

On V_2 in S_5 the ω-plane which represents the bundle T is transformed into a surface of order seven which meets the ω-plane in a curve of order four, the representation on V_2 of K_4 of S_3.

Again, a linear complex of lines in S_3 is transformed into a line-complex of order seven, and the two complexes have in common a congruence of order four. On V_2 in S_5 the intersection of

* The number of triple points of a ruled surface of order n, genus, p, is $(n-4)\ [(n-2)(n-3)/6-p]$. See Edge, loc. cit., p. 31.

V_2 with a general hyperplane is transformed into the intersection of V_2 with a hypersurface of order seven. The invariant points form a surface of order four.

CASE 2. Let us now consider the involution defined among the lines of S_3 (and hence among the points of V_2 in S_5) by the transversals of one regulus of a quadric surface and the lines of two flat pencils whose bases and whose vertices are not conjugate as to the quadric. The bases are not tangent to, nor do the vertices lie on the quadric.

Choose as the quadric surface

$$z_1^2 - z_2^2 + z_3^2 - z_4^2 = 0,$$

and on this the regulus

$$\frac{z_1 + z_2}{z_3 + z_4} = -\frac{z_3 - z_4}{z_1 - z_2} = k,$$

and as the bases of the pencils choose the planes

$$z_1 - 2z_2 = 0;\ z_2 - 2z_3 = 0,$$

and as their vertices the points (2,1,0,0); (0,2,1,0) respectively.

The involution defined by this system is

$$x_i = \phi_i(y), \qquad (i = 1, \cdots, 6), \tag{9}$$

where the ϕ_i are of degree five in y_j. The involution is rational.

In S_5, (9) defines a point transformation, involutorial and of order five. The ρ-planes which represent the plane fields α_i of S_3 and the ω-planes representing the bundles A_i are transformed into the intersection of V_2 with certain quadric hypersurfaces, and the points representative of the lines $\alpha_1\alpha_2$, A_1A_2 of S_3 are transformed into ordinary quadric surfaces on V_2. Also, the conic which represents the second regulus of the quadric surface with which we started in S_3 is transformed into the intersection of V_2 with a quartic hypersurface.

By the same method employed in Case 1, the locus of invariant points is found to be of order three and to contain the conic representative of the second regulus.

A general line on V_2 is transformed into a rational curve of order five meeting the line in three points. A general plane of either system is transformed into a surface of order five on V_2

meeting the plane in a cubic curve, and a general linear three-dimensional locus on V_2 is transformed into a three-dimensional locus of order five meeting the linear locus in a cubic surface.

CASE 3. The involution defined by the transversals of two generators from one regulus of each of two quadric surfaces which have no generator in common is of order three with an invariant complex of order two. Among the invariant lines are the generators of the other regulus on each quadric.

The transformation on V_2 has for fundamental elements the two conics representative of the second reguli of the quadrics in S_3 and a ruled surface R_4 of order four, each of whose points is transformed into the entire generator of R_4 on which it lies. The surface R_4 is the representation in S_5 of the (4,4) congruence of lines of S_3 formed by ∞^1 pencils whose vertices are on the curve of intersection of the two quadrics and whose bases are the planes determined by the intersecting generators of the reguli considered on the two quadrics.

A general line on V_2 is transformed into a rational curve of order three meeting the line in two points. A general plane of either system is transformed into a cubic surface which meets the plane in a conic, and a general linear three-dimensional locus is transformed into a three-dimensional locus of order three meeting the linear locus in a quadric surface.

Other properties of this transformation have also been obtained by A. R. Williams* by a partly different method; they will not be repeated here.

CORNELL UNIVERSITY

---

* Williams, *The transformation of lines of space by means of two quadratic reguli*, the present issue of this Bulletin, vol. 38 (1932), p. 554.

NOTE ON A THEOREM DUE TO BROMWICH

BY H. L. GARABEDIAN

The following well known theorem is due to Bromwich.*

THEOREM. *Suppose* (i) *that the series* $\sum a_n$ *is summable by Cesàro means of order* k *to the sum* s, (ii) *that* v_n *is a function of* x *with the properties*

$$\left.\begin{aligned} &(\alpha) && \sum n^k \left| \Delta^{k+1} v_n \right| < K\dagger \\ &(\beta) && \lim_{n\to\infty} n^k v_n = 0 \end{aligned}\right\} \quad \textit{if} \quad x > 0,$$

$$(\gamma) \qquad \lim_{x\to 0} v_n = 1,$$

where K *is independent of* x *and* n. *Then the series* $\sum a_n v_n$ *converges if* x *is positive, and*

$$\lim_{x\to 0} \sum a_n v_n = s.$$

I propose to establish this theorem by a more direct and shorter method than that used by Bromwich. Moreover, this proof affords a method of exhibiting a k-fold summability with infinite matrix of reference, analogous to well known definitions of summability with finite matrices of reference which make use of repeated means, for any v_n which satisfies the conditions of the theorem under discussion.

By hypothesis the series $\sum a_n$ is summable by Cesàro means of order k, so that if

$$S_n^{(k)} = \binom{n+k-1}{k-1} s_0 + \binom{n+k-2}{k-1} s_1 + \cdots + \binom{k-1}{k-1} s_n$$

and

$$A_n^{(k)} = \binom{n+k}{k},$$

* Mathematische Annalen, vol. 65 (1907–08), pp. 350–369; p. 359.

† Since all of the terms in the series $\sum n^k \left| \Delta^{k+1} v_n \right|$ are positive, this condition implies the convergence of the series.

then

$$C_n^{(k)} = \frac{S_n^{(k)}}{A_n^{(k)}}$$

has a definite limit s as n tends to ∞. We may also define $S_n^{(k)}$ by means of the identities

$$\sum S_n^{(k)} x^n = (1 - x)^{-k} \sum s_n x^n = (1 - x)^{-(k+1)} \sum a_n x^n,$$

from which it follows that

$$\sum s_n x^n = (1 - x)^k \sum S_n^{(k)} x^n$$

and

$$\sum a_n x^n = (1 - x)^{k+1} \sum S_n^{(k)} x^n.$$

It results at once from the last identity that

$$(1) \qquad \begin{aligned} a_n = S_n^{(k)} &- \binom{k+1}{1} S_{n-1}^{(k)} + \binom{k+1}{2} S_{n-2}^{(k)} - \cdots \\ &+ (-1)^{k+1} \binom{k+1}{k+1} S_{n-k-1}^{(k)}, \end{aligned}$$

where it is understood that when a negative subscript occurs in the formula, the corresponding $S^{(k)}$ is to be replaced by zero.

Now, we form the series

$$F(x) = \sum_{n=0}^{\infty} a_n v_n(x),$$

or, using (1),

$$(2) \quad F(x) = \sum_{n=0}^{\infty} \left[S_n^{(k)} - \binom{k+1}{1} S_{n-1}^{(k)} + \binom{k+1}{2} S_{n-2}^{(k)} - \cdots + (-1)^{k+1} \binom{k+1}{k+1} S_{n-k-1}^{(k)} \right] v_n(x).$$

We are justified by the conditions (ii) in ordering the terms of (2) with respect to $S_n^{(k)}$ to get

$$F(x) = \sum_{n=0}^{\infty} S_n^{(k)} \left[v_n(x) - \binom{k+1}{1} v_{n+1}(x) + \cdots + (-1)^r \binom{k+1}{r} v_{n+r}(x) + \cdots + (-1)^{k+1} v_{n+k+1}(x) \right],$$

or

$$F(x) = \sum_{n=0}^{\infty} \binom{n+k}{k} \Delta^{k+1} v_n(x) C_n^{(k)}. \tag{3}$$

Since $\lim_{n\to\infty} C_n^{(k)} = s$, it remains to show that the method of summation with infinite matrix of reference defined by (3) is *regular*,* which is to say that $\lim_{x\to 0} F(x) = s$. Accordingly, we must require in the present case that

$$\text{(a)} \qquad \lim_{x\to 0} \binom{n+k}{k} \Delta^{k+1} v_n(x) = 0 \text{ for every } n,$$

$$\text{(b)} \qquad \lim_{x\to 0} \sum_{n=0}^{\infty} \binom{n+k}{k} \Delta^{k+1} v_n(x) = 1,$$

$$\text{(c)} \qquad \sum_{n=0}^{\infty} \binom{n+k}{k} \left| \Delta^{k+1} v_n(x) \right| < K$$

for every $x > 0$, K independent of x.

It follows from condition (α) of the hypotheses that

$$\lim_{x\to 0} \binom{n+k}{k} \Delta^{k+1} v_n(x) = \binom{n+k}{k} \Delta^{k+1} 1 = 0.$$

Accordingly, the requirement (a) is satisfied.

Now, we need the identity

$$\sum_{n=0}^{n} \binom{n+k}{k} \Delta^{k+1} v_n(x) = v_k(x) + \sum_{\nu=0}^{k-1} \frac{k!}{(k-\nu)!\nu!} \Delta^{k-\nu} v_\nu(x) - \sum_{\nu=0}^{k} \frac{(n+k-\nu)!}{(k-\nu)!n!} \Delta^{k-\nu} v_{n+1}(x). \dagger \tag{4}$$

* See Carmichael, *The theory of summable series*, this Bulletin, vol. 25 (1918–19), pp. 97–131; p. 117.

† See H. L. Garabedian, Annals of Mathematics, vol. 32 (1930), pp. 83–106; p. 91.

As n tends to infinity every term involving n on the right-hand side of (4) tends to zero by virtue of condition (β) of the hypotheses. It follows that

$$(5) \quad \sum_{n=0}^{\infty} \binom{n+k}{k} \Delta^{k+1} v_n(x) = v_k(x) + \sum_{\nu=0}^{k-1} \frac{k!}{(k-\nu)!\nu!} \Delta^{k-\nu} v_\nu(x).$$

Moreover, by virtue of condition (α),

$$(6) \quad \lim_{x \to 0} \sum_{\nu=0}^{k-1} \frac{k!}{(k-\nu)!\nu!} \Delta^{k-\nu} v_\nu(x) = 0.$$

Accordingly, from (5), (6), and (α) we have

$$\lim_{x \to 0} \sum_{n=0}^{\infty} \binom{n+k}{k} \Delta^{k+1} v_n(x) = \lim_{x \to 0} v_k(x) = 1,$$

and the requirement (b) is fulfilled.

Finally, we note that the expression

$$\sum_{n=0}^{\infty} \binom{n+k}{k} \left| \Delta^{k+1} v_n(x) \right|$$

will be uniformly bounded or fail to be uniformly bounded according as the expression $\sum_{n=0}^{\infty} n_k \left| \Delta^{k+1} v_n(x) \right|$ is uniformly bounded or fails to be uniformly bounded. It is understood in this statement that x is restricted to positive values. Hence, by condition (γ), the last of the requirements (c) for regularity is satisfied.

We conclude that

$$\lim_{x \to 0} F(x) = \lim_{x \to 0} \sum_{n=0}^{\infty} \binom{n+k}{k} \Delta^{k+1} v_n(x) C_n^{(k)} = s.$$

We exhibit in the function

$$\phi(n, x) = \binom{n+k}{k} \Delta^{k+1} v_n(x)$$

a convergence factor which, associated with a method of summation with infinite matrix of reference, affords a method of constructing a k-fold method of summability with infinite matrix of reference for any $v_n(x)$ which satisfies the conditions of Bromwich's theorem. Examples of functions $v_n(x)$ which satisfy these requirements* are the LeRoy convergence factor:

* H. L. Garabedian, loc. cit.

$$v_n(x) = \frac{\Gamma[(1-x)n+1]}{\Gamma(1+n)};$$

the Mittag-Leffler convergence factor:

$$v_n(x) = \frac{1}{\Gamma(1+nx)};$$

and the Dirichlet series convergence factors:

$$v_n(x) = e^{-\lambda(n)x},$$

where $\lambda(n)$ must be a logarithmico-exponential function of n which tends to infinity with n but not as slowly as $\log n$ nor faster than n^{Δ}, where Δ is any constant however large.

NORTHWESTERN UNIVERSITY

A THEOREM ON SYMMETRIC DETERMINANTS

BY W. V. PARKER

1. *Introduction.* In a recent paper* the writer proved the following theorem.

If $D=|a_{ij}|$ is a real symmetric determinant of order n, $n>5$, in which $a_{ii}=0$, $(i=1, 2, \cdots, n)$, and M is any principal minor of D of order $n-1$, then if all fourth order principal minors of M are zero, D vanishes.

The purpose of the present note is to establish a second theorem of a similar nature which applies to complex as well as to real determinants. It will be shown also that when a_{ij}, $(i\neq j)$, $(i,j=1,2, \cdots, n)$, is real and different from zero the conditions of this second theorem imply those of the above.

2. *A Second Theorem.* The theorem with which this note is concerned may be stated as follows.

THEOREM. *If $D=|a_{ij}|$ is a symmetric determinant of order n, $n>5$, in which $a_{ii}=0$, $(i=1, 2, \cdots, n)$, and M is any principal minor of D or order $n-1$, then if all fourth order principal minors of D, which are not minors of M, are zero, D vanishes.*

---

* This Bulletin, vol. 38 (1932), p. 259.

There is no loss of generality in assuming that M is the minor in the lower right hand corner of D and we will make this assumption throughout. The theorem is trivial if all elements of the first row are zero. We will assume, therefore, that there is at least one element in the first row which is not zero and we may take this element to be a_{12}.

We will first show that the theorem is true for $n=6$ and $n=7$ and then show that the general theorem follows from these two special cases. For convenience we will write each element as the square of a number which may be either real or complex.

3. *The Case* $n=6$. For this case we may write

$$(1) \qquad D=\begin{vmatrix} 0 & \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & \epsilon^2 \\ \alpha^2 & 0 & a^2 & b^2 & c^2 & d^2 \\ \beta^2 & a^2 & 0 & x^2 & y^2 & z^2 \\ \gamma^2 & b^2 & x^2 & 0 & u^2 & v^2 \\ \delta^2 & c^2 & y^2 & u^2 & 0 & w^2 \\ \epsilon^2 & d^2 & z^2 & v^2 & w^2 & 0 \end{vmatrix},$$

where $\alpha \neq 0$. Let us denote by M_{ij} the fourth-order principal minor of the determinant D obtained by striking out the ith and jth rows and the ith and jth columns. We then have, by hypothesis, $M_{ij}=0$, $(i=2, 3, 4, 5)$, $(j=3, 4, 5, 6)$, $(i<j)$. Since

$$M_{23}=(\gamma w+\delta v+\epsilon u)\,(\gamma u+\delta v-\epsilon u)\,(\gamma w-\delta v+\epsilon u)\,(\gamma w-\delta v-\epsilon u)=0,$$

either $\gamma^2w^2=(\epsilon u+\delta v)^2$ or $\gamma^2w^2=(\epsilon u-\delta v)^2$. Similar relations are obtained from the fact that the other minors are zero.

If now we subtract β^2/α^2 times the second column from the third, γ^2/α^2 times the second column from the fourth, δ^2/α^2 times the second column from the fifth, and ϵ^2/α^2 times the second column from the sixth, and then transform the rows in the same way, we get

$$(2) \qquad D=-(1/\alpha^4)\begin{vmatrix} -2\beta^2a^2 & \alpha^2x^2-\beta^2b^2-\gamma^2a^2 & \alpha^2y^2-\delta^2a^2-\beta^2c^2 & \alpha^2z^2-\beta^2d^2-\epsilon^2a^2 \\ \alpha^2x^2-\beta^2b^2-\gamma^2a^2 & -2\gamma^2b^2 & \alpha^2u^2-\gamma^2c^2-\delta^2b^2 & \alpha^2v^2-\gamma^2d^2-\epsilon^2b^2 \\ \alpha^2y^2-\delta^2a^2-\beta^2c^2 & \alpha^2u^2-\gamma^2c^2-\delta^2b^2 & -2\delta^2c^2 & \alpha^2w^2-\delta^2d^2-\epsilon^2c^2 \\ \alpha^2z^2-\beta^2d^2-\epsilon^2a^2 & \alpha^2v^2-\gamma^2d^2-\epsilon^2b^2 & \alpha^2w^2-\delta^2d^2-\epsilon^2c^2 & -2\epsilon^2d^2 \end{vmatrix}.$$

Since $M_{34}=M_{35}=M_{36}=M_{45}=M_{46}=M_{56}=0$, if any one of the quantities a, b, c, d, β, γ, δ, ϵ is zero, every element in the row and column of (2) containing this quantity will be zero and

hence $D=0$. In a similar way we may show that when $M_{ij}=0$, $(i=2, 3, 4, 5)$, $(j=3, 4, 5, 6)$, $(i<j)$, D vanishes if any element not in the principal diagonal is zero.

Let us now consider the case when all elements outside of the principal diagonal are different from zero. The computation here is simplified somewhat by assuming $\alpha=\beta=\gamma=\delta=\epsilon=1$ and this may be done without loss of generality.*

Since $M_{34}=M_{35}=M_{36}=M_{45}=M_{46}=M_{56}=0$, we may write

$$D = -16a^2b^2c^2d^2\Delta, \tag{3}$$

where

$$\Delta = \begin{vmatrix} -1 & \pm 1_x & \pm 1_y & \pm 1_z \\ \pm 1_x & -1 & \pm 1_u & \pm 1_v \\ \pm 1_y & \pm 1_u & -1 & \pm 1_w \\ \pm 1_z & \pm 1_v & \pm 1_w & -1 \end{vmatrix}, \tag{4}$$

where $\pm 1_x$ is $+1$ or -1 according as, in M_{56},

$$x^2=(a+b)^2 \quad \text{or} \quad x^2=(a-b)^2,$$

and similarly for the others.

Since a, b, c, d are all different from zero, D vanishes if and only if Δ vanishes. Since $M_{45}=M_{46}=M_{56}=0$, one of the following must be true:

(I)

	1	2	3	4	5	6	7	8
x	+	−	+	+	+	−	−	−
y	+	+	−	+	−	+	−	−
z	+	+	+	−	−	−	+	−

,

---

* $D=\alpha^4\beta^4\gamma^4\delta^4\epsilon^4 D_1$, where

$$D_1 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2/(\alpha^2\beta^2) & b^2/(\alpha^2\gamma^2) & c^2/(\alpha^2\delta^2) & d^2/(\alpha^2\epsilon^2) \\ 1 & a^2/(\alpha^2\beta^2) & 0 & x^2/(\beta^2\gamma^2) & y^2/(\beta^2\delta^2) & z^2/(\beta^2\epsilon^2) \\ 1 & b^2/(\alpha^2\gamma^2) & x^2/(\beta^2\gamma^2) & 0 & u^2/(\gamma^2\delta^2) & v^2/(\gamma^2\epsilon^2) \\ 1 & c^2/(\alpha^2\delta^2) & y^2/(\beta^2\delta^2) & u^2/(\gamma^2\delta^2) & 0 & w^2/(\delta^2\epsilon^2) \\ 1 & d^2/(\alpha^2\epsilon^2) & z^2/(\beta^2\epsilon^2) & v^2/(\gamma^2\epsilon^2) & w^2/(\delta^2\epsilon^2) & 0 \end{vmatrix}.$$

Since α, β, γ, δ, ϵ are all different from zero, D and D_1 vanish simultaneously and so do their corresponding minors.

where 1 means $x^2=(a+b)^2$, $y^2=(a+c)^2$, $z^2=(a+d)^2$, and similarly for the others.

Since $M_{34}=M_{35}=M_{36}=0$, one of the following must be true:

(II)

	1′	2′	3′	4′	5′	6′	7′	8′
u	+	−	+	+	+	−	−	−
v	+	+	−	+	−	+	−	−
w	+	+	+	−	−	−	+	−

,

where 1′ means $u^2=(b+c)^2$, $v^2=(b+d)^2$, $w^2=(c+d)^2$, and similarly for the others. In substituting these values, it is sufficient to take only positive square roots since D and all minors are even functions of their arguments.

If now we indicate the fact that i and j' are both true by (i, j'), we find that $M_{23}=M_{24}=M_{25}=M_{26}=0$ is impossible for all cases except (1, 8′); (2, 4′); (3, 3′); (4, 2′); (5, 4′); (6, 3′); (7, 2′); (8, 8′). In each of these cases Δ is zero and hence D vanishes. The theorem is true, therefore, for $n=6$.

4. *The Case* $n=7$. In this case we write

$$D=\begin{vmatrix} 0 & \alpha^2 & \beta^2 & \gamma^2 & \lambda^2 & \mu^2 & \nu^2 \\ \alpha^2 & 0 & a^2 & b^2 & c^2 & d^2 & f^2 \\ \beta^2 & a^2 & 0 & m^2 & n^2 & p^2 & q^2 \\ \gamma^2 & b^2 & m^2 & 0 & x^2 & y^2 & z^2 \\ \lambda^2 & c^2 & n^2 & x^2 & 0 & u^2 & v^2 \\ \mu^2 & d^2 & p^2 & y^2 & u^2 & 0 & w^2 \\ \nu^2 & f^2 & q^2 & z^2 & v^2 & w^2 & 0 \end{vmatrix},$$

where $\alpha\neq 0$. Denote by M_{ijk} the fourth order principal minor of D obtained by striking out the ith, jth, and kth rows and columns. Then by hypothesis $M_{ijk}=0$, $(i=2, 3, 4, 5)$, $(j=3, 4, 5, 6)$, $(k=4, 5, 6, 7)$, $(i<j<k)$. Each of these minors may be written as the product of four factors as in the previous case.

By a transformation similar to the one above and making use of the fact that $M_{ijk}=0$, $(i=3, 4, 5)$, $(j=4, 5, 6)$, $(k=5, 6, 7)$, $(i<j<k)$, we find that D is zero if any element, outside the principal diagonal, in the first two rows is zero. Similarly we may

show that when $M_{ijk}=0$, $(i=2, 3, 4, 5)$, $(j=3, 4, 5, 6)$, $(k=4, 5, 6, 7)$, $(i<j<k)$, D vanishes if any element outside the principal diagonal is zero.

Let us suppose now that all elements outside of the principal diagonal are different from zero. As before we may assume $\alpha=\beta=\gamma=\lambda=\mu=\nu=1$ and we get

$$D = -32a^2b^2c^2d^2f^2\Delta, \tag{5}$$

where

$$\Delta = \begin{vmatrix} -1 & \pm 1_m & \pm 1_n & \pm 1_p & \pm 1_q \\ \pm 1_m & -1 & \pm 1_x & \pm 1_y & \pm 1_z \\ \pm 1_n & \pm 1_x & -1 & \pm 1_u & \pm 1_v \\ \pm 1_p & \pm 1_y & \pm 1_u & -1 & \pm 1_w \\ \pm 1_q & \pm 1_z & \pm 1_v & \pm 1_w & -1 \end{vmatrix},$$

where $\pm 1_m$ is $+1$ or -1 according as, in M_{567}, $m^2=(a+b)^2$ or $m^2=(a-b)^2$, and similarly for the others.

Since a, b, c, d, f are all different from zero, D vanishes if and only if Δ vanishes. Since $M_{356}=M_{357}=M_{367}=0$, we must have one of the cases under (I) above, where by 1 in (I) we now mean $x^2=(b+c)^2$, $y^2=(b+d)^2$, $z^2=(b+f)^2$ and similarly for the others. Since $M_{345}=M_{346}=M_{347}=0$, we must have one of the cases under (II) above, where by $1'$ in (II) we now mean $u^2=(c+d)^2$, $v^2=(c+f)^2$, $w^2=(d+f)^2$, and similarly for the others.

As before, we will indicate the fact that i and j' are both true by (i, j'). We find that $M_{234}=M_{235}=M_{236}=M_{237}=0$ is impossible for every case except $(1, 8')$; $(2, 4')$; $(3, 3')$; $(4, 2')$; $(5, 4')$; $(6, 3')$; $(7, 2')$; $(8, 8')$, and in each of these cases Δ is zero and hence D vanishes. The theorem is true, therefore, for $n=7$.

The theorem for the general case now follows immediately. If there is an element in the first row which is not zero, there is a second order principal minor, not a minor of M, which is not zero. If now all third order principal minors which are not minors of M are zero, D is of rank two.* If there is a third order principal minor, not a minor of M, which is not zero, D is of rank three provided that all fifth order principal minors which are not minors of M are zero. If there is a fifth order principal

* Bôcher, *Introduction to Higher Algebra*, p. 57, Theorem 1.

minor, not a minor of M, which is not zero, it follows from the two special cases above that D is of rank five.* If, therefore, there is an element in the first row which is not zero, D is of rank five or less.

If all elements outside of the principal diagonal are different from zero, then all fifth order principal minors of D, which are not minors of M, are zero and hence D is of rank three. It will be sufficient to show that the fifth order minor, K, in the upper left hand corner of (1) is zero when the elements outside of the principal diagonal are all different from zero. K is zero if and only if the third order minor, Δ', in the upper left of (4) is zero. But $\Delta' = 2[1+(\pm 1_x)(\pm 1_y)(\pm 1_u)]$ and hence is zero in each of the cases, (1, 8′); (2, 4′); (3, 3′); (4, 2′); (5, 4′); (6, 3′); (7, 2′); (8, 8′), for which $M_{23} = M_{24} = M_{25} = M_{26} = 0$ is possible.

That this may not be true when there are two zeros in the same rows is shown by the following example. Let

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & a^2 & (a+b)^2 & b^2 & c^2 \\ 1 & a^2 & 0 & b^2 & (a+b)^2 & c^2 \\ 1 & (a+b)^2 & b^2 & 0 & a^2 & c^2 \\ 1 & b^2 & (a+b)^2 & a^2 & 0 & c^2 \\ 0 & c^2 & c^2 & c^2 & c^2 & 0 \end{vmatrix},$$

where a, b, and $a+b$ are all real and different from zero. All fourth order principal minors of D except those that are minors of the fifth order minor in the lower right, are zero. The fifth order minor in the upper left hand corner of D is not zero, however.†

MISSISSIPPI WOMAN'S COLLEGE

* Bôcher, loc. cit.

† L. M. Blumenthal, this Bulletin, vol. 37 (1931), p. 755.

QUADRATIC PARTITIONS: PAPER II

BY E. T. BELL

1. *Equivalent Identities.** We refer to I for the general notation, and now define a function F of 4 arguments by

$$F(w, z, u, v) \equiv F((z, u, v) \mid w) = -F(w, z, -v, -u);$$

that is, F is of the parity indicated and is arbitrary beyond the stated condition.

Obviously, $F(w, z, u, v)$ is an instance of the completely arbitrary parity function $f((z, u, v) \mid w)$ having parity $p(3 \mid 1)$, and $f((z, u, v) \mid w) - f((z, -v, -u) \mid w)$ is an instance of $F(w, z, u, v)$. Let $\sum_i$ refer to a set of values of (w, z, u, v). Then, by what precedes,

$$\sum_i c_i F(w_i, z_i, u_i, v_i) = 0,$$

$$\sum_i c_i [f((z_i, u_i, v_i) \mid w_i) - f((z_i, -v_i, -u_i) \mid w_i)] = 0$$

are both true or both false. If both are true, and hence if either is true, the relations are called *equivalent*. Many instances of equivalent identities will occur in these notes. The equivalence is always, as above, immediate from the properties of the functions implied in the parity notation, and need not be further discussed.

2. ϑ, ϕ *Identities*. Combined with the transformation of the second order, Jacobi's formula for the product of 4 ϑ-functions gives an endless chain of identities between functions ϑ, ϕ, which can be written down by elementary algebra.† These lead to identities between parity functions in any number of arguments summed over quadratic partitions. The first of these ϑ, ϕ identities to be discussed is

* The preceding note, this Bulletin, vol. 37 (1931), pp. 870–5, will be cited as I.

† At the writer's request, Mr. W. H. Gage of Victoria College, B.C., has derived a large number of these identities. A particular identity can easily be checked independently; Mr. Gage's systematic derivations reveal the simple interconnections between all. A sufficient account of the method will doubtless appear elsewhere.

$$\Phi(w, z, u, v) + \Phi(w, -z, -u, -v) - \Phi(w, -z, v, u) - \Phi(w, z, -v, -u) \equiv 0,$$

where $\Phi(w, z, u, v)$ denotes

$$\phi_{001}\left(\frac{w+z}{2}, u\right)\vartheta_3\left(\frac{v-z}{2}, q^{1/2}\right)\vartheta_3\left(\frac{w-u}{2}, q^{1/2}\right).$$

The required expansions in connection with this are

$$\vartheta_3(x, q^{1/2}) = \sum q^{\nu^2} \cos 2\nu x,$$
$$\phi_{001}(x, y) = \csc y + 4\sum q^n\left[\sum \sin(2tx + \tau y)\right].$$

3. *Passage to Parity Functions.* The above ϑ, ϕ identity implies and is implied by that obtained by equating to zero the coefficient of q^n, namely,

$$\begin{aligned} &4\sum \sin(t+\nu_2)w\left[\cos\left\{(t-\nu_1)z + (\tau-\nu_2)u + \nu_1 v\right\}\right. \\ &\quad \left. - \cos\left\{(t-\nu_1)z - \nu_1 u - (\tau-\nu_2)v\right\}\right] \\ &\quad + \epsilon_2(n)\sum \sin bw\left[\csc u \sin(az + bu - av)\right. \\ &\quad \left. - \csc v \sin(az + au - bv)\right] = 0, \end{aligned}$$

the summations referring to the partitions $n = 2t\tau + \nu_1^2 + \nu_2^2$, $n = a^2 + b^2$, where $\epsilon_2(n) = 1$ or 0 according as n is or is not a sum of 2 squares. Refer to the first identity in I, §7. Then we pass at once to

$$\begin{aligned} &4\sum F(t+\nu_2, t-\nu_1, \tau-\nu_2, \nu_1) \\ &= \epsilon_2(n)\sum \operatorname{sgn} b\left[\left\{1 - e(b)\right\}F(b, a, a, 0) + 2\sum_{r=1}^{B} F(b, a, a, 2r - e(b))\right], \end{aligned}$$

with F as in §1, and $B \equiv [\frac{1}{2}|b|]$.

4. *Form for F Entire in* (u, v). Denote by $F_1(w, z, u, v)$ the function obtained from $F(w, z, u, v)$ by restricting the latter to be entire in (u, v), and apply to the trigonometric identity in §3 the first formula in the last pair of I, §6. Then

$$4\sum F_1(t+\nu_2, t-\nu_1, \tau-\nu_2, \nu_1) = \epsilon_2(n)\sum F_1(b, a, a, \rho'(b)).$$

In the next note we shall indicate how this may be written down at once from §3, end.

5. *Contraction.* We describe the general process for $f(\xi_1, \cdots, \xi_r | \eta_1, \cdots, \eta_s)$ as in I, §2. This function has the same parity as

$$f_1(\xi_p, \cdots, \xi_t | \,) f(\xi_1, \cdots, \xi_r | \, \eta_1, \cdots, \eta_s),$$

where $p, \cdots, t$ are distinct integers of the set $1, \cdots, r$, and f_1 is arbitrary beyond the parity implied in the notation. The *order* of a vector is the number of its coordinates. Let $P, \cdots, T$ be the respective orders of $\xi_p, \cdots, \xi_t$. Let $\{N\}$ denote for the moment the zero vector of order N. Choose for f_1 the function defined by

$$f_1(\{P\}, \cdots, \{T\} | \,) = 1,$$
$$f_1(\xi_p, \cdots, \xi_t | \,) = 0, (\xi_p, \cdots, \xi_t) \neq (\{P\}, \cdots, \{T\}).$$

Let ξ_i^h, η_j^h, where h denotes a superscript, be a set of values of ξ_i, η_j. Then, from a relation of the type

$$\sum_h c_h f(\xi_1^h, \cdots, \xi_r^h | \, \eta_1^h, \cdots, \eta_s^h) = 0,$$

we infer the like with $\xi_p^h, \cdots, \xi_t^h$ suppressed. For, the product $f_1 f$ is an instance of f.

Let ξ_i^h, η_j^h refer to a partition, say π. Then, from π and $\xi_p^h = \{P\}, \cdots, \xi_t^h = \{T\}$, at least one of the indeterminates can be eliminated. There results a new partition in fewer indeterminates, restricted by all the conditions which express the arithmetical character (sign, integrality, evenness or oddness, etc.) of the eliminated indeterminates. The $\sum_h$ with $\xi_p^h, \cdots, \xi_t^h$ suppressed thus refers to the new or *contracted* partition, and we say that $f(\xi_1, \cdots, \xi_r | \eta_1, \cdots, \eta_s)$ has been *contracted with respect to* $\xi_p, \cdots, \xi_t$.

6. *Application of* §5 *to* §§3, 4. We contract $F(w, z, u, v)$ with respect to z*. The partitions in §3 contract to

$$n = xy + z^2, \quad n = t^2,$$

the first of which is restricted,

$$x > 0, \; y - x > 0, \; y - x \equiv 2 \bmod 4, \; z \gtreqless 0,$$

and t in the second, by the general notation in I, §3, is >0. The function $F(w, z, u, v)$ in §3 is replaced by

$$G(w, u, v) \equiv G((u, v) | \, w) = -G((-v, -u) | \, w),$$

* There can be no confusion between the argument z in F and z in the partition, as the partition is defined by the conditions stated.

and we have

$$\sum G\left(x+z, \frac{y-x}{2}-z, x\right) = \epsilon_1(n) \sum_{r=1}^{T} G(t, 0, 2r-e(t)),$$

where $\epsilon_1(n)=1$ or 0 according as n is or is not a square >0, and $T=[t/2]$.

Similarly, from §4, if $G_1(w, u, v)$ is $G(w, u, v)$ with the restriction of entirety in (u, v) we get

$$4\sum G_1\left(x+z, \frac{y-x}{2}-z, x\right) = \epsilon_1(n)\left[G_1(t, 0, \rho'(t)) - G_1(t, 0, \rho'(-t))\right].$$

CALIFORNIA INSTITUTE OF TECHNOLOGY

THE TRANSFORMATION OF LINES OF SPACE BY MEANS OF TWO QUADRATIC REGULI*

BY A. R. WILLIAMS

If we take two quadratic reguli, a line l meets two generators of each. To l we make correspond the other transversal of the four generators. This involutory transformation of the lines of space is one of three, quite similar in principle.† This case admits a very simple and effective algebraic treatment without the use of hyperspace.

We may take for the equations of two non-singular quadrics with real rulings $x_1^2+x_2^2-x_3^2-x_4^2=0$ and $a^2x_1^2+b^2x_2^2-c^2x_3^2-d^2x_4^2=0$. On the former lies the regulus R_1 defined by $x_1-x_3=m(x_4-x_2)$, $x_1+x_3=1/(m(x_4+x_2))$. The Plücker coordinates of a line of this regulus are

$$(1)\quad p_{12}:p_{13}:p_{14}:p_{23}:p_{42}:p_{34} = (m^2+1):2m:(m^2-1):(m^2-1):2m:-(m^2+1).$$

The other regulus R_1' on the same quadric is given by the

---

* Presented to the Society, November 28, 1932.

† Discussed by Mr. J. M. Clarkson in the present issue of this Bulletin, vol. 38, pp. 533–540.

equation $x_1-x_3=m'(x_4+x_2)$, $x_1+x_3=1/(m'(x_4-x_2))$. The coordinates of a line of this regulus are, in the same order,

$$(1') \quad (m'^2+1):-2m':-(m'^2-1):(m'^2-1):2m':(m'^2+1).$$

Similarly, on the other quadric lies the regulus R_2, a general line of which has the coordinates

$$(2)\ cd(n^2+1):bd\cdot 2n:bc(n^2-1):ad(n^2-1):ac\cdot 2n:-ab(n^2+1);$$

and the regulus R_2', a general line of which has the coordinates

$$(2') \quad cd(n'^2+1):-bd\cdot 2n':-bc(n'^2-1)$$
$$:ad(n'^2-1):ac\cdot 2n':ab(n'^2+1).$$

To effect the transformation we shall use the reguli R_1 and R_2. That is, to any line l will correspond l', where l' meets the same generators of R_1 and R_2 as l does. A general line p_{ik} meets two generators of R_1 whose parameters are determined by

$$(3) \quad m^2(-p_{12}+p_{14}+p_{23}+p_{34})$$
$$+2m(p_{13}+p_{42})-p_{12}-p_{14}-p_{23}+p_{34}=0.$$

Similarly, p_{ik} meets two generators of R_2 whose parameters are given by

$$(4) \quad n^2(-abp_{12}+adp_{14}+bcp_{23}+cdp_{34})+2n(acp_{13}+bdp_{42})$$
$$-abp_{12}-adp_{14}-bdp_{23}+cdp_{34}=0.$$

If any other line p'_{ik} is to meet the same generators of R_1 and R_2, we have, by comparing coefficients, two sets of equations which immediately reduce to

$$(5) \quad \frac{p'_{12}-p'_{34}}{p_{12}-p_{34}}=\frac{p'_{13}+p'_{42}}{p_{13}+p_{42}}=\frac{p'_{14}+p'_{23}}{p_{14}+p_{23}},$$

$$(6) \quad \frac{abp'_{12}-cdp'_{34}}{abp_{12}-cdp_{34}}=\frac{acp'_{13}+bdp'_{42}}{acp_{13}+bdp_{42}}=\frac{adp'_{14}+bcp'_{23}}{adp_{14}+bcp_{23}}.$$

Introducing the proportionality factors ρ and σ, we have $p'_{12}-p'_{34}=\rho(p-{}_{12}p_{34})$, $abp'_{12}-cdp'_{34}=\sigma(abp_{12}-cdp_{34})$ and two

similar pairs of equations. Solving these equations, we have

$$
\begin{aligned}
p'_{12} &= \frac{-\rho cd(p_{12}-p_{34})+\sigma(abp_{12}-cdp_{34})}{ab-cd},\\
p'_{34} &= \frac{-\rho ab(p_{12}-p_{34})+\sigma(abp_{12}-cdp_{34})}{ab-cd},\\
p'_{13} &= \frac{-\rho bd(p_{13}+p_{42})+\sigma(acp_{13}+bdp_{42})}{ac-bd},\\
p'_{42} &= \frac{\rho ac(p_{13}+p_{42})-\sigma(acp_{13}+bdp_{42})}{ac-bd},\\
p'_{14} &= \frac{-\rho bc(p_{14}+p_{23})+\sigma(adp_{14}+bcp_{23})}{ad-bc},\\
p'_{23} &= \frac{\rho ad(p_{14}+p_{23})-\sigma(adp_{14}+bcp_{23})}{ad-bc}.
\end{aligned}
\tag{7}
$$

The ratio $\rho:\sigma$ is found from the condition that $p'_{12}p'_{34}+p'_{13}p'_{42}+p'_{14}p'_{23}=0$. This gives $A\rho^2-B\rho\sigma+C\sigma^2=0$ where

$$A=abcd\frac{(p_{12}-p_{34})^2}{(ab-cd)^2}-\frac{(p_{13}+p_{42})^2}{(ac-bd)^2}-\frac{(p_{14}+p_{23})^2}{(ad-bc)^2}$$

and

$$C=\frac{(abp_{12}-cdp_{34})^2}{(ab-cd)^2}-\frac{(acp_{13}+bdp_{42})^2}{(ac-bd)^2}-\frac{(adp_{14}+bcp_{23})^2}{(ad-bc)^2}.$$

Evidently one value of $\rho:\sigma$ is 1:1. Remembering that we may write $p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23}=0$, we verify easily that $A-B+C=0$. Hence the other root is

$$\rho:\sigma = C:A. \tag{8}$$

Thus the coordinates of p'_{ik} are cubic functions of those of p_{ik}.

Equations (7) show that a line p_{ik} for which $\rho=\sigma\neq 0$ is invariant; that is, it meets two generators of R_1 and two of R_2 which have only one distinct transversal. The condition $\rho=\sigma$ gives the quadratic complex

$$-\frac{abp_{12}^2-cdp_{34}^2}{ab-cd}+\frac{acp_{13}^2+bdp_{42}^2}{ac-bd}+\frac{adp_{14}^2+bcp_{23}^2}{ad-bc}=0. \tag{9}$$

The generators of R_1 and R_2 belong to this complex. To a line p_{ik} for which $\rho=0$ but $\sigma\neq 0$ corresponds a line whose coordi-

nates, down to a constant factor, are

$$p'_{12}=p'_{34}=\frac{abp_{12}-cdp_{34}}{ab-cd},$$

$$p'_{13}=-p'_{42}=\frac{acp_{13}+bdp_{42}}{ac-bd},$$

$$p'_{14}=-p'_{23}=\frac{adp_{14}+bcp_{23}}{ad-bc}.$$

Since $-p'_{12}+p'_{14}+p'_{23}+p'_{34}=0$, $p'_{13}+p'_{42}=0$, and $-p'_{12}-p'_{14}-p'_{23}+p'_{34}=0$, this line by equation (3) meets all the generators of R_1; that is, it is itself a generator of R_1'. It is geometrically obvious that if p_{ik} meets two generators of R_2 which are met by the same generator of R_1', the latter will be p_{ik}'. Any generator of R_1' meets two generators of R_2; and no two generators of R_1' meet the same two generators of R_2 if the quadrics are not specially related. Hence the complex $\rho=0$ is made up of ∞^1 congruences, each congruence consisting of the lines that meet a pair of generators of R_2 that are met by the same generator of R_1'. Each generator of R_2 is a member of two such pairs. Similarly, to a line p_{ik} for which $\sigma=0$ but $\rho\neq 0$ corresponds a generator of R_2'; and the complex $\sigma=0$ consists of ∞^1 congruences, each composed of the lines that meet a pair of generators of R_1 that are met by the same generator of R_2'. The generators of both R_1' and R_2' satisfy $\rho=0$ and $\sigma=0$. That is, they belong to the congruence of lines that do not have unique transforms. Such lines are easily found. Through P, any point on the intersection of the two quadrics, pass g_1, a generator of R_1, and g_2, a generator of R_2; also g_1' and g_2' generators of R_1' and R_2' respectively. The plane of g_1' and g_2' contains f_1 and f_2, generators of R_1 and R_2 respectively. Thus any line through P in this plane meets the four generators g_1 and f_1 of R_1, and g_2 and f_2 of R_2. Analytically, if we take a_{ik} and b_{ik}, any two generators of R_1' and R_2' that are supposed to meet, see (1′) and (2′), we find easily that the coordinates of any line of their pencil $a_{ik}+\lambda b_{ik}$ satisfy $\rho=\sigma=0$. Thus the congruence is composed of ∞^1 flat pencils, each determined by a generator of R_1' and a generator of R_2'. Each such generator belongs to two of these pencils. If two generators of R_1 and two of R_2 were linearly dependent, they would lie all four on the same proper hyperboloid, or three of them would belong to the same flat pen-

cil. In either case they would have an infinite number of transversals and no one of these transversals would then have a unique transform. This cannot happen, however, unless there is some special relation between the two quadrics. For if, using (1) and (2), we let m_1 and m_2 be the parameters of two generators of R_1, and n_1 and n_2 the parameters of two generators of R_2, linear dependence would necessitate the simultaneous vanishing of all the four-rowed determinants of a matrix 6 by 4. It turns out that this can not be accomplished by variation of the parameters alone.

The characteristics of the transformation can be inferred from the above. To a ruled surface of order n corresponds one of order $3n$. For each line of the congruence $\rho=\sigma=0$, that is not at the same time a generator of R_1' or R_2', contained by a ruled surface, the order of the transform will be reduced by one. Each generator of R_1' or R_2' will reduce the order by two. Thus a generic flat pencil contains two invariant lines, two lines belonging to $\rho=0$, and two belonging to $\sigma=0$. Hence, the corresponding cubic surface contains two generators of R_1' and two of R_2', and their presence agrees with the fact that to the cubic surface corresponds the original pencil. In this connection an interesting fact may be noted. If a_{ik} and b_{ik} are two lines meeting at O, any line of their pencil is given by $a_{ik}+\lambda b_{ik}$. Since the coordinates of the transform of the latter contain λ to the third degree, it will for all values of λ meet two fixed lines, generally distinct, whose coordinates are functions of a_{ik} and b_{ik}. One of these must pass through O and the other must lie in the plane of the pencil. Since the pencil contains two invariant lines, the former, passing through O, is the double line of the cubic surface and the latter the directrix. Calling the double line r and the directrix r', it is easy to see that to any plane pencil whose center is on r and whose plane passes through r' corresponds a similar cubic surface whose double line is r and directrix r'. This relation is reciprocal. To any plane pencil whose center is on r' and whose plane passes through r corresponds a ruled cubic surface whose double line is r' and whose directrix is r. That is, if any line meets r and r', its transform by (7) does so also, and the congruence determined by r and r' is invariant as a whole. Here we have, as it were, a transformation within a transformation. Taking a general line for r, is there a unique r' corresponding to it in this sense? To answer this, let r be determined by the points

(y) and (z) so that $r_{ik}=y_iz_k-y_kz_i$. Then we find that in addition to the quadric cone of invariant lines whose center is (y) there is a flat pencil of lines with center at (y) whose transforms by (7) meet r. There is a similar flat pencil at (z) and at every other point of r. The planes of these flat pencils form an axial pencil whose axis is r'. Its coordinates are easily found in terms of the coordinates of r and prove to be just the coordinates of the line which would correspond to r by the transformation exactly similar to (7) which is effected by means of the supplementary reguli R_1' and R_2'. As we should expect, this latter transformation admits the same invariant complex as the original. In fact, if r in the above discussion is an invariant line under (7), the flat pencil of lines at any point of r whose transforms by (7) meet r contains r. To such a flat pencil corresponds a ruled cubic surface of which r is both double line and directrix.

To a quadratic regulus corresponds a ruled sextic. If the regulus contains for example two generators of R_1' and one of R_2', the order of the corresponding surface should reduce to nothing. This is obviously correct. For if the intersection of two quadrics consists in part of two skew lines, the remainder must be two skew lines of the other ruling. So all the rulings of the regulus in question, including the generator of R_2' which belongs to it, meet the same two rulings of R_1. Therefore, to the whole regulus corresponds the generator of R_2'. To a linear complex corresponds a cubic complex which contains the generators of R_1' and R_2' each counted ∞^1 times. For consider a pair of generators of R_2 that are met by the same generator of R_1'. The lines of the linear complex that meet this pair of generators constitute a quadratic regulus, and to every line of it corresponds the generator of R_1'. To such a cubic complex corresponds the complex of order 9 made up of ρ^2, σ^2, and the original linear complex. Finally, it should be noted that the order of the complex corresponding to a complex that contains R_1' or R_2' is lowered by 2. Thus the harmonic complex, that is, the quadratic complex of lines that meet the two quadrics in four harmonic points, contains both R_1' and R_2', and to it corresponds another quadratic complex. Of course, R_1 and R_2 also belong to the harmonic complex; and R_1' and R_2' belong to its transform by (7); but R_1 and R_2 do not.

THE UNIVERSITY OF CALIFORNIA

ON THE TRIGONOMETRIC DEVELOPMENTS OF CERTAIN DOUBLY PERIODIC FUNCTIONS OF THE SECOND KIND*

BY M. A. BASOCO

1. *Introduction.* The class of meromorphic functions which satisfy periodicity relations of the form

$$f(z + 2\omega_1) = c_1 f(z), \quad f(z + 2\omega_2) = c_2 f(z), \tag{1}$$

where the multipliers c_1 and c_2 are independent of z, and ω_1/ω_2 is a complex number with non-vanishing imaginary part, has been named by Hermite† *doubly periodic of the second kind.* It is possible to make the study of these functions depend on others of the same type, but such that one of the multipliers, say c_1, is unity. In what follows we shall assume, further, that the periods $(2\omega_1, 2\omega_2)$ are $(\pi, \pi\tau)$, where $\tau = a + ib$, $b > 0$.

Particular interest is attached to the functions (2) below, which belong to the category just defined. In terms of the Jacobi theta functions they have the form

$$\phi_{\alpha\beta\gamma}(x, y) = \vartheta_1' \frac{\vartheta_\alpha(x + y)}{\vartheta_\beta(x)\vartheta_\gamma(y)}, \tag{2}$$

where x, y are independent complex variables, and α, β, γ are certain triads, sixteen in number, which can be selected from the numbers 0, 1, 2, 3. These functions were first discovered by Jacobi‡ and have been studied by Kronecker§, Hermite‖, Teixeira¶ and others. More recently, E. T. Bell** has pointed out their importance in connection with certain results in number

* Presented to the Society, April 8, 1932.

† Hermite, Comptes Rendus, vol. 85 (1877), . . . vol. 94 (1882); Annales de l'École Normale Supérieure, (3), vol. 2, 1885, p. 303. Oeuvres, vol. IV, p. 190–200.

‡ Jacobi, Werke, vol. 2, pp. 291–351.

§ Kronecker, Monatsberichte der Berliner Akademie, 1881, pp. 1165–1172; Werke, vol. IV, pp. 309, 318.

‖ Loc. cit.; see also Lerch, Acta Mathematica, vol. 12 (1889), pp. 51–55.

¶ Teixeira, Journal für Mathematik, vol. 125 (1901), pp. 301–318.

** E. T. Bell, Transactions of this Society, vol. 22 (1921); Colloquium Series of this Society, vol. 7, p. 88. Giornale di Matematiche, vol. 59 (1921).

theory; in these arithmetical applications, their explicit Fourier expansions play a fundamental rôle.

In what follows, we shall be primarily concerned with deriving the Fourier developments for the squares of the $\phi_{\alpha\beta\gamma}(x, y)$. However, our analysis is perfectly general and the expansion for any positive integral power of these functions may be calculated if desired. It is on account of the possible arithmetic applications that we have carried out the calculations for the squares; these expansions turn out to have a relatively simple structure, while the higher powers are of a far greater complexity and hence may not yield simple and interesting arithmetic results.

The analysis used originates with Teixeira (loc. cit.); however, for our purpose, it was necessary to generalize somewhat certain of his results, as we proceed to indicate.

2. *Extension of Teixeira's Results.* Let $f(z)$ be a doubly periodic function of the second kind with periods equal to π and $\pi\tau$, the corresponding multipliers being unity and c, respectively, where c is independent of z. Suppose that in a fundamental period cell $f(z)$ has k poles, $z=a_r$, $r=1, 2, 3, \cdots, k$. Further, let the order of these poles be m_r respectively, so that in the neighborhood of $z=a_r$, $f(z)$ has the Laurent expansion

$$(3)\quad f(z) = \frac{A_{m_r}^{(r)}}{(z-a_r)^{m_r}} + \cdots + \frac{A_2^{(r)}}{(z-a_r)^2} + \frac{A_1^{(r)}}{z-a_r} + P(z-a_r).$$

Consider a parallelogram $pqrs$ consisting of $\beta+1$ cells above the real axis and α below. Inside this parallelogram, $f(z)$ has $\alpha+\beta+1$ poles which have the affixes

$$(4)\qquad z = a_r + n\pi\tau = \omega_{r,n},$$

($r=1, 2, 3 \cdots k$; $n=-\alpha+1, \cdots, -1, 0, 1, 2, \cdots, \beta$). The corresponding residues are $c^n A_1^{(r)}$.

Next, consider the auxiliary function given by

$$(5)\qquad \phi(t) \equiv \frac{e^{2it}}{e^{2it}-e^{2iz}} f(t) = \tfrac{1}{2}\left[1 - i \operatorname{ctn}(t-z)\right] f(t).$$

If we apply Cauchy's theorem to the contour integral $\int_{(C)} \phi(t)dt$ the contour C being the boundary of the parallelogram $pqrs$, we obtain, after certain reductions* which we shall omit, the follow-

* For more details on the nature of these reductions we refer the reader to some analogous calculations which are indicated in a paper by the writer in this Bulletin, vol. 37 (1931), pp. 117–124.

ing result which is valid in a strip parallel to the axis of reals and bounded by lines of which rq and sp are segments:

$$(6)\quad \begin{cases} f(z) = \sum_{m=-\infty}^{\infty} C_m e^{2miz} - i \sum_{(r,n)} A_1^{(r)} c^n (1 + i \operatorname{ctn}(z - \omega_{r,n})) \\ \qquad + \sum_{(r,n)} \sum_{s=1}^{m_r-1} \frac{A_{s+1}^{(r)}}{s!} c^n D^{(s)}_{\omega_{r,n}} \operatorname{ctn}(z - \omega_{r,n}), \end{cases}$$

where

$$(7)\quad \begin{cases} C_m = \dfrac{c^{-\alpha} q^{2m\alpha}}{1 - cq^{-2m}} \sum_{s=1}^{m_r} \sum_{r=1}^{k} \dfrac{(-1)^s (2i)^s m^{s-1} A_s^{(r)}}{(s-1)!} e^{-2mia_r}, \\ q = e^{\pi i \tau}, \qquad (m = 1, 2, 3, \cdots), \\ C_{-m} = \dfrac{c^{1+\beta} q^{2m(1+\beta)}}{1 - cq^{2m}} \sum_{s=1}^{m_r} \sum_{r=1}^{k} \dfrac{(2i)^s m^{s-1} A_s^{(r)}}{(s-1)!} e^{2mia_r}, \\ C_0 = \dfrac{2ic^{-\alpha}}{1 - c} \sum_{r=1}^{k} A_1^{(r)}, \end{cases}$$

and $D^{(s)}$ is the differential operator of order s, and $\omega_{r,n}$ is the argument $a_r + n\pi\tau$ which is to be substituted after the differentiation has been performed.

3. *Application to the Functions* $\Phi_{\alpha\beta\gamma}(z, v)$. We shall use the notation $\Phi_{\alpha\beta\gamma}(z, v)$ to denote the squares of the $\phi_{\alpha\beta\gamma}(z, v)$ defined by equation (2). These separate into two groups (A) and (B) according as their poles, qua functions of z, are congruent to $\pi\tau/2$ or $\pi\tau$. They all come under the case where $k = 1$ and $m_r = 2$, (that is, there is but one pole $z = a$ in a period cell and its order is two). With these values and also with (α,β) equal to (0, 0) our fundamental formula (6) reduces to the following:

$$(8)\quad \begin{cases} f(z) = 2iA_1 \left[\sum_{n=0}^{\infty} \dfrac{e^{2ni(z-a)}}{1 - cq^{-2n}} + \sum_{n=1}^{\infty} \dfrac{cq^{2n} e^{-2ni(z-a)}}{1 - cq^{2n}} \right] \\ \qquad + 4A_2 \left[\sum_{n=1}^{\infty} \dfrac{ne^{2ni(z-a)}}{1 - cq^{-2n}} - \sum_{n=1}^{\infty} \dfrac{ncq^{2n} e^{-2ni(z-a)}}{1 - cq^{2n}} \right] \\ \qquad - iA_1(1 + i \operatorname{ctn}(z - a)) + A_2 \csc^2(z - a). \end{cases}$$

We shall now transform this expression into a form suitable for the calculation of the arithmetical expansions of the functions in group (A). To do this, use is made of the relationships

$$(9)\qquad \begin{cases} 1 + i\,\mathrm{ctn}\,(z-a) = 2\sum_{n=0}^{\infty} e^{2ni(z-a)}, \\ \csc^2(z-a) = -4\sum_{n=1}^{\infty} n e^{2ni(z-a)}, \end{cases}$$

which are valid provided $I(z) > I(a)$. Furthermore, as may be seen from the properties of the theta functions, the $\Phi_{\alpha\beta\gamma}(z, v)$ as functions of z have the multipliers unity and $c = e^{-4iv}$.

Substituting these values in (8) we find, after some reduction,

$$(10)\qquad f(z) = A_1\psi(z, v, a) + A_2 D_a^{(1)}\psi(z, v, a),$$

$$(11)\qquad \psi(z, v, a) = e^{-2iv}\sum_{n=-\infty}^{\infty} \frac{e^{2ni(z-a-\pi\tau/2)}}{\sin(2v + n\pi\tau)}.$$

In order to obtain an expansion suitable for use in connection with the functions of group (B), all that is necessary is to rewrite (8) in the form

$$(12)\qquad f(z) = A_1\zeta(z, v, a) + A_2 D_a^{(1)}\zeta(z, v, a),$$

$$(13)\qquad \begin{aligned} \zeta(z, v, a) = {} & \mathrm{ctn}\,(z-a) + \mathrm{ctn}\,2v + e^{2iv}\sum_{n=1}^{\infty} \frac{q^n e^{2ni(z-a)}}{\sin(2v + n\pi\tau)} \\ & + e^{-2iv}\sum_{n=1}^{\infty} \frac{q^n e^{-2ni(z-a)}}{\sin(2v - n\pi\tau)}. \end{aligned}$$

4. *The Expansions in Group* (A). Consider the function

$$(14)\qquad f(z) \equiv \Phi_{001}(z, v) = \vartheta_1'^2 \frac{\vartheta_0^2(z+v)}{\vartheta_0^2(z)\vartheta_1^2(v)};$$

its periods are π and $\pi\tau$ with multipliers unity and e^{-4iv} respectively. We may select the fundamental period cell so that $a = -\pi\tau/2$ is the affix of the pole to be used in (10). Further, a slight calculation shows that the coefficients A_1 and A_2 of the principal part of its Laurent expansion about $z = -\pi\tau/2$ have the values $A_1 = 2e^{2iv}\vartheta_1'(v)/\vartheta_1(v)$ and $A_2 = e^{2iv}$. Equations (10) and (11) then yield the result

$$\Phi_{001}(z, v) = \vartheta_1'^2 \frac{\vartheta_0^2(z+v)}{\vartheta_0^2(z)\vartheta_1^2(v)} = \frac{2\vartheta_1'(v)}{\vartheta_1(v)} \sum_{n=-\infty}^{\infty} \frac{e^{2niz}}{\sin(2v + n\pi\tau)}$$

(15)

$$- 2i \sum_{n=-\infty}^{\infty} \frac{ne^{2niz}}{\sin(2v + n\pi\tau)}.$$

In order to reduce this expression to arithmetical form, we use the following relations, which may easily be proved:

(16) $$\csc(2v + n\pi\tau) = -2i \sum_{n=1}^{\infty} q^{nm} e^{2imv}, \quad (m = 1, 3, 5, 7, \cdots),$$

(17) $$\csc(2v - n\pi\tau) = 2i \sum_{m=1}^{\infty} q^{nm} e^{-2imv}.$$

These hold simultaneously and for all positive integral n in case $\frac{1}{2}I(\pi\tau) < I(v) < \frac{1}{2}I(\pi\tau)$. The expansions sought will be readily obtained from (15) if each sum is replaced by two others in which the indices of summation range from $n=1$ to $n=\infty$, if we pair off corresponding terms and make use of (16) and (17). Thus we obtain, on changing our notation (z, v) to (x, y) in order to be in agreement with current use, the development

(I)
$$\begin{cases} \Phi_{001}(x, y) = \vartheta_1'^2 \dfrac{\vartheta_0^2(x+y)}{\vartheta_0^2(x)\vartheta_1^2(y)} = \dfrac{2\vartheta_1'(y)}{\vartheta_1(y)} \left[\dfrac{1}{\sin 2y} \right. \\ \qquad\qquad + 4 \sum q^n \Big(\sum \sin 2(tx + \tau y) \Big) \Big] \\ \qquad\qquad - 8 \sum q^n \Big(\sum t \cos 2(tx + \tau y) \Big), \end{cases}$$

in which the summation appearing as the coefficient of q^n is to be extended over all the positive integral divisors t, τ of n, τ being always odd.

Replacing (x, y) successively by $(x, y+\pi/2)$, $(x+\pi/2, y)$, $(x+\pi/2)$, $(y+\pi/2)$ in the preceding, we obtain the following:

(II)
$$\begin{cases} \Phi_{202}(x, y) = \vartheta_1'^2 \dfrac{\vartheta_2^2(x+y)}{\vartheta_0^2(x)\vartheta_2^2(y)} = -2 \dfrac{\vartheta_2'(y)}{\vartheta_2(y)} \left[\dfrac{1}{\sin 2y} \right. \\ \qquad\qquad + 4 \sum q^n \Big(\sum \sin 2(tx + \tau y) \Big) \Big] \\ \qquad\qquad + 8 \sum q^n \Big(\sum t \cos 2(tx + \tau y) \Big); \end{cases}$$

$$\text{(III)}\quad \left\{\begin{aligned}\Phi_{331}(x,\,y) &= \vartheta_1'^2\,\frac{\vartheta_3^2(x+y)}{\vartheta_3^2(x)\vartheta_1^2(y)} = 2\frac{\vartheta_1'(y)}{\vartheta_1(y)}\Big[\frac{1}{\sin 2y}\\ &\quad + 4\sum(-1)^n q^n\Big(\sum \sin 2(tx+\tau y)\Big)\Big]\\ &\quad - 8\sum(-1)^n q^n\Big(\sum t\cos 2(tx+\tau y)\Big);\end{aligned}\right.$$

$$\text{(IV)}\quad \left\{\begin{aligned}\Phi_{032}(x,\,y) &= \vartheta_1'^2\,\frac{\vartheta_0^2(x+y)}{\vartheta_3^2(x)\vartheta_2^2(y)} = -2\frac{\vartheta_2'(y)}{\vartheta_2(y)}\Big[\frac{1}{\sin 2y}\\ &\quad + 4\sum(-1)^n q^n\Big(\sum \sin 2(tx+\tau y)\Big)\Big]\\ &\quad + 8\sum(-1)^n q^n\Big(\sum t\cos 2(tx+\tau y)\Big).\end{aligned}\right.$$

In an entirely similar manner, we may derive from

$$f(z) = \Phi_{100}(z,\,v) = \vartheta_1'^2\,\frac{\vartheta_1^2(z+v)}{\vartheta_0^2(z)\vartheta_0^2(v)}$$

the following developments:

$$\text{(V)}\quad \left\{\begin{aligned}\Phi_{100}(x,\,y) &= \vartheta_1'^2\,\frac{\vartheta_1^2(x+y)}{\vartheta_0^2(x)\vartheta_0^2(y)} = 2\frac{\vartheta_0'(y)}{\vartheta_0(y)}\Big[\frac{1}{\sin 2y}\\ &\quad + 4\sum q^n\Big(\sum \sin 2(tx+\tau y)\Big)\Big]\\ &\quad - 8\sum q^n\Big(\sum t\cos 2(tx+\tau y)\Big);\end{aligned}\right.$$

$$\text{(VI)}\quad \left\{\begin{aligned}\Phi_{203}(x,\,y) &= \vartheta_1'^2\,\frac{\vartheta_2^2(x+y)}{\vartheta_0^2(x)\vartheta_3^2(y)} = -2\frac{\vartheta_3'(y)}{\vartheta_3(y)}\Big[\frac{1}{\sin 2y}\\ &\quad + 4\sum q^n\Big(\sum \sin 2(tx+\tau y)\Big)\Big]\\ &\quad + 8\sum q^n\Big(\sum t\cos 2(tx+\tau y)\Big);\end{aligned}\right.$$

$$\text{(VII)}\quad \left\{\begin{aligned}\Phi_{230}(x,\,y) &= \vartheta_1'^2\,\frac{\vartheta_2^2(x+y)}{\vartheta_3^2(x)\vartheta_0^2(y)} = 2\frac{\vartheta_0'(y)}{\vartheta_0(y)}\Big[\frac{1}{\sin 2y}\\ &\quad + 4\sum(-1)^n q^n\Big(\sum \sin 2(tx+\tau y)\Big)\Big]\\ &\quad - 8\sum(-1)^n q^n\Big[\sum t\cos 2(tx+\tau y)\Big];\end{aligned}\right.$$

$$\text{(VIII)}\quad \left\{\begin{aligned}\Phi_{133}(x,\,y) &= \vartheta_1'^2\,\frac{\vartheta_1^2(x+y)}{\vartheta_3^2(x)\vartheta_3^2(y)} = -2\frac{\vartheta_3'(y)}{\vartheta_3(y)}\Big[\frac{1}{\sin 2y}\\ &\quad + 4\sum(-1)^n q^n\Big(\sum \sin 2(tx+\tau y)\Big)\Big]\\ &\quad + 8\sum(-1)^n q^n\Big(\sum t\cos 2(tx+\tau y)\Big).\end{aligned}\right.$$

5. *The Expansion in Group* (B). To obtain the Fourier series for the functions in this group we return to formulas (12) and (13). If the series (16) and (17) are used in connection with the expression defining the function $\zeta(z, v, a)$ we obtain the Fourier expansion for this function. Thus,

$$(18)\quad \zeta(z, v, a) = \operatorname{ctn} 2v + \operatorname{ctn}(z - a) + 4\sum q^n\Big(\sum \sin 2(d(z - a) + \delta v)\Big)$$

where the sum which appears as the coefficient of q^n ranges over all divisors d, δ of n such that δ is even. We may note, in passing, that $\zeta(z, v, a) \equiv \phi_{111}(z-a, 2v)$. Consider, now, the function

$$f(z) \equiv \Phi_{111}(z, v) = \vartheta_1'^2 \frac{\vartheta_1^2(z + v)}{\vartheta_1^2(z)\vartheta_1^2(v)};$$

all that is necessary to obtain the development of $f(z)$ is to calculate the coefficients A_1, A_2 relative to the pole $a=0$. We find that $A_2=1$ and $A_1=2\vartheta_1'(v)/\vartheta_1(v)$. Hence on applying (12) and (18) and changing, as before, the notation (z, v) to (x, y), we have

$$(IX)\quad \begin{cases} \Phi_{111}(x, y) = \vartheta_1'^2 \dfrac{\vartheta_1^2(x + y)}{\vartheta_1^2(x)\vartheta_1^2(y)} = 2\dfrac{\vartheta_1'(y)}{\vartheta_1(y)}\Big[\operatorname{ctn} x + \operatorname{ctn} 2y \\ \qquad + 4\sum q^n\Big(\sum \sin 2(dx + \delta y)\Big)\Big] \\ \qquad + \dfrac{1}{\sin^2 x} - 8\sum q^n\Big(\sum d \cos 2(dx + \delta y)\Big). \end{cases}$$

From this, the following are deduced:

$$(X)\quad \begin{cases} \Phi_{221}(x, y) = \vartheta_1'^2 \dfrac{\vartheta_2^2(x + y)}{\vartheta_2^2(x)\vartheta_1^2(y)} = 2\dfrac{\vartheta_1'(y)}{\vartheta_1(y)}\Big[-\tan x + \operatorname{ctn} 2y \\ \qquad + 4\sum q^n\Big(\sum(-1)^d \sin 2(dx + \delta y)\Big)\Big] \\ \qquad + \dfrac{1}{\cos^2 x} - 8\sum q^n\Big(\sum(-1)^d d\cos 2(dx+\delta y)\Big); \end{cases}$$

$$(XI)\quad \begin{cases} \Phi_{212}(x, y) = \vartheta_2'^2 \dfrac{\vartheta_2^2(x + y)}{\vartheta_1^2(x)\vartheta_2^2(y)} = 2\dfrac{\vartheta_2'(y)}{\vartheta_2(y)}\Big[\operatorname{ctn} x + \operatorname{ctn} 2y \\ \qquad + 4\sum q^n\Big(\sum \sin 2(dx + \delta y)\Big)\Big] \\ \qquad + \dfrac{1}{\sin^2 x} - 8\sum q^n\Big(\sum d \cos 2(dx + \delta y)\Big); \end{cases}$$

$$(\text{XII})\quad \begin{cases} \Phi_{122}(x, y) = \vartheta_1'^2 \dfrac{\vartheta_1^2(x+y)}{\vartheta_2^2(x)\vartheta_2^2(y)} = 2\dfrac{\vartheta_2'(y)}{\vartheta_2(y)}[-\tan x + \operatorname{ctn} 2y \\ \qquad + 4\sum q^n(\sum(-1)^d \sin 2(dx+\delta y))] \\ \qquad + \dfrac{1}{\cos^2 x} - 8\sum q^n(\sum d(-1)^d \cos 2(dx+\delta y)). \end{cases}$$

In an analogous manner, if we let

$$f(z) = \Phi_{010}(z, v) = \vartheta_1'^2 \frac{\vartheta_0^2(z+v)}{\vartheta_1^2(z)\vartheta_0^2(v)},$$

we obtain the following:

$$(\text{XIII})\quad \begin{cases} \Phi_{010}(x, y) = \vartheta_1'^2 \dfrac{\vartheta_0^2(x+y)}{\vartheta_1^2(x)\vartheta_0^2(y)} = 2\dfrac{\vartheta_0'(y)}{\vartheta_0(y)}[\operatorname{ctn} x + \operatorname{ctn} 2y \\ \qquad + 4\sum q^n(\sum \sin 2(dx+\delta y))] \\ \qquad + \dfrac{1}{\sin^2 x} - 8\sum q^n(\sum d \cos 2(dx+\delta y)); \end{cases}$$

$$(\text{XIV})\quad \begin{cases} \Phi_{320}(x, y) = \vartheta_1'^2 \dfrac{\vartheta_3^2(x+y)}{\vartheta_2^2(x)\vartheta_0^2(y)} = 2\dfrac{\vartheta_0'(y)}{\vartheta_0(y)}[-\tan x + \operatorname{ctn} 2y \\ \qquad + 4\sum q^n(\sum(-1)^d \sin 2(dx+\delta y))] \\ \qquad + \dfrac{1}{\cos^2 x} - 8\sum q^n(\sum(-1)^d d\cos 2(dx+\delta y)); \end{cases}$$

$$(\text{XV})\quad \begin{cases} \Phi_{313}(x, y) = \vartheta_1'^2 \dfrac{\vartheta_3^2(x+y)}{\vartheta_1^2(x)\vartheta_3^2(y)} = 2\dfrac{\vartheta_3'(y)}{\vartheta_3(y)}[\operatorname{ctn} x + \operatorname{ctn} 2y \\ \qquad + 4\sum q^n(\sum \sin 2(dx+\delta y))] \\ \qquad + \dfrac{1}{\sin^2 x} - 8\sum q^n(\sum d \cos 2(dx+\delta y)); \end{cases}$$

$$(\text{XVI})\quad \begin{cases} \Phi_{023}(x, y) = \vartheta_1'^2 \dfrac{\vartheta_0^2(x+y)}{\vartheta_2^2(x)\vartheta_3^2(y)} = 2\dfrac{\vartheta_3'(y)}{\vartheta_3(y)}[-\tan x + \operatorname{ctn} 2y \\ \qquad + 4\sum q^n(\sum(-1)^d \sin 2(dx+\delta y))] \\ \qquad + \dfrac{1}{\cos^2 x} - 8\sum q^n(\sum(-1)^d d\cos 2(dx+\delta y)). \end{cases}$$

6. *Conclusion.* The developments obtained in what precedes contain terms which are multiplied by expressions of the form $\vartheta_\alpha'(y)/\vartheta_\alpha(y)$; the arithmetized trigonometric expansions for these are well known* and may be substituted in our developments; when the multiplication and reduction are carried out the corresponding coefficient of the general power of q will then be a finite sum extended over the solutions of certain quaternary quadratic forms.

It should, perhaps, be noticed that our series lead in one direction to the expansions for the squares of the Jacobian elliptic functions, while in another they give rise to several doubly periodic functions of the *third kind*. Thus, for example, if in expansion (I) we let y tend to the value $\pi/2$ we obtain the development for the square of the delta amplitude function:

$$\vartheta_0^2\vartheta_3^2\frac{\vartheta_3^2(x)}{\vartheta_0^2(x)} = -\frac{\vartheta_2''}{\vartheta_2} + 8\sum q^n\Big(\sum t\cos 2tx\Big), \tag{19}$$

where

$$-\frac{\vartheta_2''}{\vartheta_2} = 1 + 8\sum\frac{q^{2n}}{(1+q^{2n})^2}.$$

Again, if in (I) we put $x=-y$ we get an expansion for $\vartheta_1'^2\vartheta_0^2/(\vartheta_0^2(x)\vartheta_1^2(x))$. If in (V) we put $x=y$ and apply the transformation of the second order we obtain a development for $\vartheta_1'\vartheta_1(x)\vartheta_2(x)\vartheta_3(x)/\vartheta_0^3(x)$. These examples suffice to indicate the possibilities in this direction.

Finally, it should be noted that the methods here used will not yield the expansions for the remaining forty-eight $\Phi_{\alpha\beta\gamma}(x,y)$. These could be obtained in a slightly modified form by multiplying each of the sixteen functions given above by a suitable elliptic function such as (19) above.†

THE UNIVERSITY OF NEBRASKA

* Bell, Messenger of Mathematics, vol. 54 (1924), p. 169. Reference should also be made to certain reduction formulas in his paper in the Giornale di Matematiche (loc. cit.).

† In this connection we may refer to a Chicago thesis (1930) by D. A. F. Robinson, which is to appear in the Proceedings of the Royal Society of Canada. In this work, expansions for the forty-eight $\phi_{\alpha\beta\gamma}(x, y)$ in modified form are given. The methods there used could, doubtless, be applied to obtain the remaining $\Phi_{\alpha\beta\gamma}(x, y)$.

QUADRATIC PARTITIONS: PAPER III

BY E. T. BELL

1. *Simple Summation Formulas.* This note is independent of partitions. It gives the general summation formulas of which a very special case might have been used to pass directly from §3 to §4 of the preceding note,* and which will be used in future. The final formulas considerably extend and generalize many in the literature of the Bernoullian and allied functions.

Let $f(x)$ be an entire function of x. Write $M \equiv [(n-1)/2]$, $N \equiv [n/2]$;

$$f_\beta(n) \equiv \sum_{r=1}^{M} f(2r - 1 + e(n)), \quad f_\gamma(n) \equiv \sum_{r=1}^{M} (-1)^r f(2r - 1 + e(n)),$$

$$f_\eta(n) \equiv \sum_{r=1}^{N} (-1)^r f(2r - e(n)), \quad f_\rho(n) \equiv \sum_{r=1}^{N} f(2r - e(n)).$$

One of the pairs (β, η), (ρ, γ) is sufficient, since

$$f_\rho(2n - 1) = f_\beta(2n) = \sum_{r=1}^{n-1} f(2r),$$

$$f_\rho(2n - 2) = f_\beta(2n - 1) = \sum_{r=1}^{n-1} f(2r - 1),$$

$$f_\gamma(2n - 1) = f_\eta(2n - 2) = \sum_{r=1}^{n-1} (-1)^r f(2r - 1),$$

$$f_\gamma(2n) = f_\eta(2n - 1) = \sum_{r=1}^{n-1} (-1)^r f(2r).$$

The like applies to the more general sums $f_{\xi p \sigma}(n)$ in §2.

It is required to express $f_\xi(n)$ $(\xi = \beta, \gamma, \eta, \rho)$ as explicit functions of n. Write

$$C_\beta(n) \equiv 2e(n)f(0) + 2f(n) + 2e(n)\{f(2B') - f(-2B')\} \cdot 2\{1 - e(n)\}\{f(R') - f(-R')\},$$

* This Bulletin, this issue (vol. 38, No. 8), pp. 551–554. The notation and definitions are given in I, ibid., vol. 37 (1931), pp. 870–875.

$$C_\gamma(n) \equiv 4e(n)f(0) - 4(-1)^M f(n) + 2\{1 - e(n)\}\{f(E) + f(-E)\}$$
$$+ 2(-1)^{e(n)}e(n)\{f(2G') - f(-2G')\},$$
$$C_\alpha(n) \equiv 2\{1 - e(n)\}f(0) + e(n)\{f(E) + f(-E)\}$$
$$- \{1 - e(n)\}\{f(2G') - f(-2G')\},$$
$$C_\rho(n) \equiv \{1 - e(n)\}f(0) + e(n)\{f(R') - f(-R')\}$$
$$+ \{1 - e(n)\}\{f(2B') - f(-2B')\}.$$

Then the required expressions are given by

$$4f_\beta(n) = f(\beta'(n)) - C_\beta(n), \quad 8f_\gamma(n) = (-1)^M f(\gamma'(n)) - C_\gamma(n),$$
$$4f_\eta(n) = (-1)^N f(\eta(n)) - C_\eta(n), \quad 2f_\rho(n) = f(\rho'(n)) - C_\rho(n).$$

It will be sufficient to indicate the genesis of one, say the last. We have (see I, §§§3, 6, 7),

$$\cos \rho'(n)x = 1 - e(n) + 2\sum_{r=1}^{N} \cos (2r - e(n))x,$$
$$\sin \rho'(n)x = 2e(n) \sin R'x + 2\{1 - e(n)\} \sin 2B'x$$
$$+ 2\sum_{r=1}^{N} \sin (2r - e(n))x,$$

from which we pass to a single identity between umbral exponentials on multiplying the second by i and adding to the first. Thence the stated formula for $2f_\rho(n)$ is written down. For, if $a, b, \cdots, c$ denote umbrae for the moment, and $A, B, \cdots, C$, x ordinaries, and if $A \exp ax + B \exp bx + \cdots + C \exp cx \equiv 0$ is an identity in x, then $Af(a) + Bf(b) + \cdots + Cf(c) = 0$. This also sufficiently indicates the proofs for the formulas in §2, which are obtained from the exponential identities just used on changing therein x into px and multiplying throughout by $\exp sx$.

2. *Extended Simple Formulas.* The arguments of f in the preceding formulas are in arithmetical progression with common difference 1. The following refer to any arithmetical progression. Let p, s be ordinates, and write

$$_{\beta p s}(n) \equiv \sum_{r=1}^{M} f(p(2r - 1 + e(n)) + s),$$

$$f_{\gamma ps}(n) \equiv \sum_{r=1}^{M}(-1)^r f(p(2r-1+e(n))+s),$$

$$f_{\eta ps}(n) \equiv \sum_{r=1}^{N}(-1)^r f(p(2r-e(n))+s),$$

$$f_{\rho ps}(n) \equiv \sum_{r=1}^{N} f(p(2r-e(n))+s).$$

These include those in §1: $f_{\xi 10}(n)=f_\xi(n)$.

Denote by $C_{\xi ps}(n)$ the result of performing the substitution $x\to px+s$ on the argument of $f(x)$ in $C_\xi(n)$. For example,

$$C_{\rho ps}(n) \equiv \{1-e(n)\}f(s)+e(n)\{f(pR'+s)-f(-pR'+s)\} + \{1-e(n)\}\{f(2pB'+s)-f(-2pB'+s)\}.$$

Then the expression of $f_{\xi ps}(n)$ as an explicit function of n is obtained from that for $f_\xi(n)$ by performing the same substitution on the argument $\xi'(n)$ of $f(\xi'(n))$ and replacing $C_\xi(n)$ by $C_{\xi ps}(n)$. For example,

$$2f_{\rho ps}(n) = f(p\rho'(n)+s) - C_{\rho ps}(n).$$

3. *Multiple Summation Formulas.* Let $f(x_1, \cdots, x_j), t>1$, be an entire function of $(x_1, \cdots, x_j)$. We write $f(x_1, \cdots, x_t)$ as a purely symbolic or ideal product of t functions $f^j(x_j)$ (j superscript) of one argument, $f(x_1, \cdots, x_j)\equiv f^1(x_1)\cdots f^1(x_j)$. By definition, in this multiplication, $f_{ij}(x_{j_1})\cdots f_{ij}(x_{j_t})\equiv f^1(x_1)\cdots f^t(x_j)$, where $j_1, \cdots, j_t$ is any permutation of $1, \cdots, t$.

Write $M_j\equiv[(n_j-1)/2]$, and let p_j, s_j be ordinaries. Then the expression of

$$2^{2t}\sum_{r_1=1}^{M_1}\cdots\sum_{r_t=1}^{M_t} f(p_1(2r_1-e(n_1))+s_1, \cdots, p_t(2r_t-e(n_t))+s_t)$$

as an explicit function of $n_1, \cdots, n_t$ is

$$\prod_{j=1}^{t}[f^j(p_j\beta'(n_j)+s_j) - C^j_{\beta p_j s_j}(n_j)],$$

in which $C^j_{\beta p_j s_j}(n_j)$ is written down from §2 by giving to each f in $C_{\beta p_j s_j}(n_j)$ as there defined the superscript j. The indicated product is distributed precisely as in a commutative ring; symbolic products of f^j's are then replaced by their equivalent functions of t arguments by the above definition.

To validate the process, proceed from $t=2$ by mathematical induction. For $t=2$, we consider $f(x_1, x_2)$ as a function of x_1, keep x_2 fixed, and apply the summation formulas of §2. In each term of the result we then consider $f(x_1, x_2)$ as a function of x_2 and apply §2.

In exactly the same way multiple summations equivalent to symbolic products (as above) of any number of factors of one or more of the types giving the explicit forms of the function $f_{\xi p^s}(n)$, $(\xi=\beta, \gamma, \eta, \rho)$, in §2 can be written out as functions of the upper limits of the summations.

By the method of proof in §§1, 2, it follows that these formulas remain true under linear transformations of the arguments of the entire functions. The like does not hold for non-linear transformations, as the product of two or more umbrae is undefined.

CALIFORNIA INSTITUTE OF TECHNOLOGY

A LOGICAL EXPANSION IN MATHEMATICS*

BY HASSLER WHITNEY†

1. *Introduction.* Suppose we have a finite set of objects, (for instance, books on a table), each of which either has or has not a certain given property A (say of being red). Let n, or $n(1)$, be the total number of objects, $n(A)$ the number with the property A, and $n(\overline{A})$ the number without the property A (with the property not-A or $\overline{A}$). Then obviously

$$n(\overline{A}) = n - n(A). \tag{1}$$

Similarly, if $n(A\ B)$ denote the number with both properties A and B, and $n(\overline{A}\ \overline{B})$ the number with neither property, that is, with both properties not-A and not-B, then

$$n(\overline{A}\,\overline{B}) = n - n(A) - n(B) + n(AB), \tag{2}$$

which is easily seen to be true.

The extension of these formulas to the general case where any number of properties are considered is quite simple, and is well

---

* Presented to the Society, October 25, 1930.

† National Research Fellow.

known to logicians. It should be better known to mathematicians also; we give in this paper several applications which show its usefulness.

The notation is that used in a paper by the author, *Characteristic functions and the algebra of logic*,* which we shall refer to as C. F. It should cause no confusion if we use the same symbols for characteristic functions, sets, and properties. The paper named is not essential for an understanding of the present paper.

2. *The Logical Expansion.* We prove now the general formula

$$
\begin{aligned}
n(\overline{A}_1\overline{A}_2\cdots\overline{A}_m) &= n - [n(A_1) + n(A_2) + \cdots + n(A_m)] \\
&+ [n(A_1A_2) + n(A_1A_3) + \cdots + n(A_{m-1}A_m)] \\
&- [n(A_1A_2A_3) + \cdots] + \cdots + (-1)^m n(A_1A_2\cdots A_m),
\end{aligned} \tag{3}
$$

which gives the number of objects *without* certain properties in terms of the numbers of objects *with* various of these properties. The formula is exactly what we should get if we multiplied out the expression in brackets in $n[(1-A_1)(1-A_2)\cdots(1-A_m)]$ by ordinary algebra and applied the general formulas

$$n(F+G) = n(F) + n(G),\ n(-F) = -n(F).$$

We assume the formula holds if there are two properties involved, and shall prove it for the case that three properties are involved. The proof obviously holds if the numbers two and three are replaced by i and $i+1$, and hence the formula is true in general, by mathematical induction.

Consider the objects counted in $n(\overline{A}_1\overline{A}_2)$, that is, those with neither of the properties A_1 and A_2. We wish to know how many of these have the property $\overline{A}_3$. Applying (1) to this set, we have

$$n(\overline{A}_1\overline{A}_2\overline{A}_3) = n(\overline{A}_1\overline{A}_2) - n(\overline{A}_1\overline{A}_2A_3).$$

We know, by hypothesis, that

$$n(\overline{A}_1\overline{A}_2) = n - [n(A_1) + n(A_2)] + n(A_1A_2).$$

To find $n(\overline{A}_1\overline{A}_2A_3)$ we need merely consider those objects with the property A_3, and apply the expansion to this set. Thus

$$n(\overline{A}_1\overline{A}_2A_3) = n(A_3) - [n(A_1A_3) + n(A_2A_3)] + n(A_1A_2A_3).$$

* To appear in the Annals of Mathematics.

Hence

$$(4)\quad n(\overline{A}_1\overline{A}_2\overline{A}_3) = n - [n(A_1) + n(A_2) + n(A_3)] + [n(A_1A_2) + n(A_1A_3) + n(A_2A_3)] - n(A_1A_2A_3),$$

as required.

3. *The Measure of Characteristic Functions.* This section relates this paper to the paper C. F. We shall find a general class of formulas which contains the logical expansion as a special case, making use of *characteristic functions*. With each element x of a set of n objects R we associate an integer $A(x)$, positive, negative, or zero. We define the *measure of* A by the equation

$$(5)\quad n(A) = \sum_{x \text{ in } R} A(x).$$

Suppose A were one for certain elements of R (which elements form the set A'), and zero for the rest. Then A is the characteristic function of A', and $n(A)$ is just the number of elements in A'. In particular, if $A \equiv 1$, that is, A' is R, then $n(A)$ is

$$(6)\quad n(1) = n,$$

and if $A \equiv 0$, that is, A' contains no elements, then $n(A)$ is

$$(7)\quad n(0) = 0.$$

If A and B are any functions and p and q are any numbers, then

$$(8)\quad n(pA + qB) = \sum_{x \text{ in } R} [pA(x) + qB(x)] = p\sum A(x) + q\sum B(x) = pn(A) + qn(B).$$

The logical expansion now follows at once if we expand $\overline{A}_1\overline{A}_2 \cdots \overline{A}_m$ into the second normal form as in C. F.*

4. *On Prime Numbers.* Let

$$p_1, p_2, \cdots, p_m$$

be any set of positive integers. We wish to find N, the number of numbers less than or equal to a given number x, which are not divisible by any of these numbers. If we let R be the set of all numbers $\leqq x$ and let P_i be those divisible by p_i, then

* Note that, for characteristic functions, $\bar{A}_i = 1 - A_i$. We make this substitution, multiply out, and use (8).

$$N = n(\overline{P}_1\overline{P}_2 \cdots \overline{P}_m)$$

$$= n - [n(P_1) + n(P_2) + \cdots] + [n(P_1P_2) + \cdots] \tag{9}$$

$$- \cdots + (-1)^m n(P_1P_2 \cdots P_m),$$

by the logical expansion (where $n = x$). This expansion is fundamental in prime number theory.* Its importance lies in the fact that the terms of the form $n(P_{i_1}P_{i_2} \cdots P_{i_k})$ are easier to calculate directly than the desired quantity.

Suppose we wish to find the number $\phi(x)$ of prime numbers $\leqq x$. Let $p_1, p_2, \cdots, p_m$ be the primes $< \sqrt{x}$. Then the primes $\leqq x$ are just those numbers $\leqq x$ which are not divisible by any of the numbers $p_1, p_2, \cdots, p_m$, except for the number 1, together with the m numbers $p_1, p_2, \cdots, p_m$. Thus

$$\phi(x) = N + m - 1. \tag{10}$$

5. *A Problem in Probability.* Suppose a pack of cards is lying in a row on a table. If we lay out another pack of cards on top of these, what is the probability that no card of the second pack will lie on the same card of the first? If a pack contains m cards, there are $m!$ arrangements of the cards in the second pack. If there are N arrangements of the second pack such that no card of the second pack falls on the same card of the first, then the required probability is $p = N/m!$. If $n(A_i)$ denotes the number of arrangements such that the ith card of the second pack falls on the same card of the first, etc., then $N = n(\overline{A}_1 \cdot \overline{A}_2 \cdots \overline{A}_m)$. If we use the logical expansion, a typical term of the result is $(-1)^k n(A_{i_1} \cdot A_{i_2} \cdots A_{i_k})$. This is the number of arrangements in which the cards numbered $i_1, i_2, \cdots, i_k$ fall on the same cards of the first pack. These k cards being fixed in position, there are $(m-k)!$ arrangements of the remaining cards. The term is thus $(-1)^k(m-k)!$. There are $\binom{m}{k}$ terms with k factors, contributing together $(-1)^k\binom{m}{k}(m-k)! = (-1)^k m!/k!$. Summing over k and dividing by $m!$, we have

$$p = \sum_{k=0}^{m} \frac{(-1)^k}{k!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^m \frac{1}{m!},$$

which is the sum of the first $m+1$ terms in the expansion of $1/e$.

* See for instance E. Landau, *Primzahlen*, Leipzig, Tuebner, pp. 67 ff.

6. *On the Number of Ways of Coloring a Graph.* Consider any set of objects $a, b, c, \cdots, f$, and any set of pairs of these objects, as $ab, bd, \cdots, cf$. We call the whole collection a *graph*, containing the *vertices* $a, b, c, \cdots, f$, and the *arcs* $ab, bd, \cdots, cf$. It can be visualized simply by letting the vertices be points in space, and letting each arc be a curve joining the two vertices involved.

Suppose we have a fixed number λ of colors at our disposal. Any way of assigning one of these colors to each vertex of the graph in such a way that any two vertices which are joined by an arc are of different colors, will be called an admissible coloring of the graph. We wish to find the number $M(\lambda)$ of admissible colorings, using λ or fewer colors.

As a special case of this general problem we have the four-color map problem: To know if we can assign to each region of a map on a sphere one of four colors in such a way that no two regions with a common boundary are of the same color, that is, to see if $M(4)>0$. Our graph (which is just the dual graph of the map) is constructed by placing a vertex in each region of the map, and joining two vertices by an arc if the corresponding regions have a common boundary. We shall deduce a formula for the number $M(\lambda)$ of ways of coloring a graph due to Birkhoff.*

If there are V vertices in the graph G, then there are λ^V possible colorings, formed by giving each vertex in succession any one of the λ colors. Let R be this set of colorings. Let A_{ab} denote those colorings with the property that a and b are of the same color, etc. Then the set of admissible colorings is

$$\overline{A}_{ab}\overline{A}_{bd}\cdots\overline{A}_{cf},$$

and the number of colorings is, if there are E arcs in the graph,

* *A determinant formula for the number of ways of coloring a map*, Annals of Mathematics, (2), vol. 14 (1912), pp. 42–46. The formula was discovered independently by the author by the method here given. See in this connection a paper by Birkhoff, *On the number of ways of coloring a map*, Proceedings of the Edinburgh Mathematical Society, (2), vol. 2 (1930), pp. 83–91; also a paper by the author, *The coloring of graphs*, to appear in the Annals of Mathematics. In regard to the four-color map problem, see references in Birkhoff's last mentioned paper, also a paper by the author, *A theorem on graphs*, Annals of Mathematics, (2), vol. 32 (1931), p. 379.

$$
\begin{aligned}
M(\lambda) &= n(\overline{A}_{ab}\overline{A}_{bd}\cdots\overline{A}_{cf}) \\
&= n - [n(A_{ab}) + n(A_{bd}) + \cdots + n(A_{cf})] \\
&\quad + [n(A_{ab}A_{bd}) + \cdots] - \cdots \\
&\quad + (-1)^E n(A_{ab}A_{bd}\cdots A_{cf}).
\end{aligned} \tag{11}
$$

With each property A_{ab} is associated an arc ab of G. In the logical expansion, there is a term corresponding to every possible combination of the properties A_{pq}; with this combination we associate the corresponding arcs, forming a *subgraph* H of G. In particular, the first term corresponds to the subgraph containing no arcs, and the last term corresponds to the whole of G. *We let H contain all the vertices of G.*

Let us evaluate a typical term, such as $n(A_{ab}A_{ad}\cdots A_{ce})$. This is the number of ways of coloring G in λ or fewer colors in such a way that a and b are of the same color, a and d are of the same color, $\cdots$, c and e are of the same color. In the corresponding subgraph H, any two vertices that are joined by an arc must be of the same color, and thus all the vertices in a single connected piece in H are of the same color. If there are p connected pieces in H, the value of this term is therefore λ^p. If there are s arcs in H, the sign of the term is $(-1)^s$. Thus

$$(-1)^s n(A_{ab}A_{ad}\cdots A_{ce}) = (-1)^s\lambda^p.$$

If there are (p, s) (this is Birkhoff's symbol) subgraphs of s arcs in p connected pieces, the corresponding terms contribute to $M(\lambda)$ an amount $(-1)^s(p, s)\lambda^p$. Therefore, summing over all values of p and s, we find the polynomial in λ:

$$M(\lambda) = \sum_{p,s}(-1)^s(p, s)\lambda^p. \tag{12}$$

Consider a subgraph H of G of s arcs, in p connected pieces. Let us define its rank i and its nullity j by the equations $i = V - p$, $j = s - i = s - V + p$. Then $p = V - i$, $s = i + j$, and putting $(p, s) = m_{V-p,s-V+p} = m_{ij}$, (thus, m_{ij} is the number of subgraphs of G of rank i, nullity j), we have

$$
\begin{aligned}
M(\lambda) &= \sum_{i,j}(-1)^{i+j}m_{ij}\lambda^{V-i} = \sum_i m_i\lambda^{V-i}, \\
m_i &= \sum_j (-1)^{i+j}m_{ij}.
\end{aligned} \tag{13}
$$

7. *The m_i in Terms of the Broken Circuits of G.* We shall find in this section an interpretation of the coefficients of $M(\lambda)$ directly in terms of properties of the graph. Consider for instance the graph G containing the vertices a, b, c, d and the arcs ab, ac, bc, bd, cd, the arcs being given in this definite order. Make a list of the *circuits* in G, naming the arcs of a circuit in the order in which they occur above. Here, the circuits are ab, ac, bc, and bc, bd, cd, and ab, ac, bd, cd. From each circuit we now form the corresponding *broken circuit* by dropping out the last arc of the circuit. The broken circuits here are ab, ac, and bc, bd, and ab, ac, bd. Then *the number $(-1)^i m_i$ is the number of subgraphs of G of i arcs which do not contain all the arcs of any broken circuit.*

To show this, we arrange the broken circuits of G in a definite order, where we put a broken circuit P_i before a broken circuit P_j if, in naming the arcs of G one by one in the given order, all the arcs of P_i are named before all those of P_j are named, otherwise, the ordering is immaterial. Suppose there are σ broken circuits, $P_1, P_2, \cdots, P_\sigma$. We now divide the subgraphs of G into $\sigma+1$ sets (some of which may be empty), putting in the first set, S_1, all those subgraphs containing all the arcs of P_1; in the second, S_2, all those not containing P_1, but containing P_2; in the third, S_3, all those containing neither P_1 nor P_2, but containing P_3; $\cdots$; in the last set, $S_{\sigma+1}$, all those containing none of these broken circuits.

Consider now all the terms in (11) corresponding to the first set of subgraphs. Suppose α_1 is the arc we dropped out of a circuit to form the first broken circuit P_1. To each subgraph in S_1 not containing α_1 corresponds a subgraph in S_1 containing α_1 and conversely, as α_1 is not in P_1. The subgraphs of S_1, and hence the corresponding terms of (11), are thus paired off. But the two terms of each pair cancel. For let H and H' be the two corresponding subgraphs. If H is in p connected pieces, so is H', as the arc α_1 joins two vertices already connected by the broken circuit P_1. The terms each contribute λ^p therefore; but they are of opposite sign, as H' contains one more arc than H.

Consider now the terms corresponding to S_2 (if there are any such). If α_2 is the arc dropped out in forming P_2, α_2 is in neither P_1 nor P_2, on account of the way we have ordered the broken circuits. Thus to each subgraph in S_2 not containing α_2 corresponds a subgraph in S_2 containing α_2, and conversely. The cor-

responding terms of (11) are thus paired off, and they cancel, exactly as before.

Continuing, we cancel all terms in S_3, S_4, $\cdots$, S_σ. We are left only with terms in $S_{\sigma+1}$, that is, those corresponding to subgraphs not containing all the arcs of any broken circuit, and none of these have been canceled.

Consider any such term containing i arcs. The corresponding subgraph H contains no circuit, as it contains no broken circuit. If we build up H arc by arc, each arc we add joins two vertices formerly not connected therefore, and the number of connected pieces is decreased by one each time. Thus the number of connected pieces in H is $V-i$, and the corresponding term contributes $(-1)^i\lambda^{V-i}$ to $M(\lambda)$. If there are l_i such subgraphs, they together contribute an amount $(-1)^i l_i\lambda^{V-i}$. Hence, summing over i, we have $P(\lambda)=\sum_i(-1)^i l_i\lambda^{V-i}$. Comparing with (13), we see that $l_i=(-1)^i m_i$, as required.

EXAMPLES. Let G contain the vertices a, b, c, and the arcs ab, ac, bc. There is one broken circuit: ab, ac. There is one subgraph of no arcs, and $m_0=1$. There are three subgraphs of a single arc, and $-m_1=3$. There are three subgraphs of two arcs; but one of them contains the broken circuit, so $m_2=2$. The subgraph of three arcs contains the broken circuit. Hence, as $V=3$,

$$M(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda - 1)(\lambda - 2).$$

This is easily verified. For we can color a in λ ways; there are $\lambda-1$ colors left for b; there are now $\lambda-2$ colors left for c.

Let G be the graph named at the beginning of this section. If a subgraph contains the last broken circuit, it contains the first also, so we can forget the last. We find

$$M(\lambda) = \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda = \lambda(\lambda - 1)(\lambda - 2)^2,$$

which again is easily verified.

PRINCETON UNIVERSITY

ON ANGLES IN CERTAIN METRIC SPACES*

BY W. A. WILSON

1. *Introduction.* In a series of articles on metrical geometry† Menger has made a study of the geometry of certain abstract metric spaces and in particular he has obtained conditions for the congruence of metric spaces with sub-sets of euclidean spaces. In a recent article‡ he suggests a system of axioms for "angle spaces" and related problems.

It will be shown in this note that a theory of angles analogous to that of euclidean space is possible for convex complete metric spaces any four points of which are congruent with four points in some euclidean space. From this certain theorems regarding tangents to simple arcs are deduced.

2. *Notation and Definitions.* A euclidean space of n dimensions will be denoted by E_n.

If a and b are two points, the symbol ab will denote the distance between them or, at times, the straight line segment joining them.

If, corresponding to a set A, there is a set A', in some E_n congruent to A, we say that A *can be imbedded* in E_n. The word congruence has its usual meaning: A is congruent to A' if there is a one to one correspondence $x \sim x'$ between the points of A and A' such that, if $x \sim x'$ and $y \sim y'$, then $xy = x'y'$. The congruence of A to A' is denoted by $A \cong A'$.

In stating a congruence between two finite sets it will be understood that the pairs of points correspond in the order written. Thus, in the congruence $a+b+c \cong a'+b'+c'$, we have $a \sim a'$, $b \sim b'$, and $c \sim c'$; the congruence $a+b+c \cong a'+c'+b'$ is a different congruence.

Likewise, if ab and $a'b'$ denote two simple arcs, $ab \cong a'b'$ if there is a one to one correspondence between their points such

* Presented to the Society, March 25, 1932.

† *Untersuchungen über allgemeine Metrik*, Mathematische Annalen, vol. 100, pp. 75–163, and vol. 103, pp. 466–501.

‡ *Some applications of point-set methods*, Annals of Mathematics, vol. 32, pp. 739–760.

that $a\sim a'$, $b\sim b'$, and the relations $x\sim x'$ and $y\sim y'$ imply that $xy=x'y'$. As in the previous paragraph the congruences $ab\cong a'b'$ and $ab\cong b'a'$ are different.

If any four points of a space Z can be imbedded in a euclidean space, we shall say for brevity that Z has the *four-point property*.* Obviously, in that case any four points can be imbedded in E_3.

The terms *convex* and *externally convex* are used in the sense of Menger. (See Annals of Mathematics, loc. cit., p. 742.)

3. *Two Lemmas.* As a preliminary we prove two lemmas which form the basis of most of the theorems following.

LEMMA 1. *Let a, b, c, and d be four points of a metric space which are congruent to four points of some euclidean space. If a', b', and c' are points in some E_3 such that $a+b+c\cong a'+b'+c'$, there is a point d' in this E_3 such that $a+b+c+d\cong a'+b'+c'+d'$.*

PROOF. By hypothesis $a+b+c+d\cong a''+b''+c''+d''$, where a'', b'', c'', and d'' lie in some E_3. If they do not lie in an E_2, they are the vertices of a tetrahedron T''. One face of this is the triangle $a''b''c''$, which is congruent to thet riangle $a'b'c'$. Hence, by euclidean geometry, T'' is congruent to a tetrahedron $T'=a'b'c'd'$ in the E_3 under discussion. This gives at once $a+b+c+d\cong a'+b'+c'+d'$. The special cases where a'', b'', c'', and d'' lie in an E_1 or in an E_2 are treated in like manner.

LEMMA 2. *Let Z be a metric space which has the four-point property. Let a and b be any two points and k be any positive constant. Then there is at most one point c for which $ac=k\cdot ab$ and* (a) $ac+cb=ab$; (b) $ab+bc=ac$; *or* (c) $ca+ab=cb$.

PROOF. In (a) $k\leqq 1$; in (b) $k\geqq 1$; and in (c) k may have any value. We prove (b), which is typical. Suppose that for some k there were two points c and d satisfying the hypotheses. Now the four points a, b, c, and d are congruent to points a', b', c', and d' in some E_3. Since $a'b'+b'c'=a'c'$, the points a', b', and c' lie on a line L. Since $a'b'+b'd'=a'd'$, the points a', b', and d' lie on a line L'. As $L\cdot L'\supset a'+b'$, the lines L and L' are identical.

* A simple example of a metric space not having the four-point property is the surface of a euclidean sphere, with the distance between any two points defined as the length of the shorter great circle arc joining them. Another example is given in §12, where the set $e+e_1+e_2+f$ cannot be imbedded in euclidean space.

Since $a'c'$ and $a'd'$ are greater than $a'b'$ and $b'c'$ or $b'd'$, respectively, by (b), it follows that c' and d' lie on $a'b'$ produced beyond b'. Since $a'c' = k \cdot a'b' = a'd'$, we have $c' = d'$. Hence $c = d$.

4. *Collinear Points.* If a and b are two points and c is a third point satisfying one of the three conditions of Lemma 2 above, we say that c is *collinear with a and b*. In other words three points are collinear if one lies between the other two.

If the space is complete and convex, there is for each k between 0 and 1 at least one point c for which $ab = ac + cb$ and $ac = k \cdot ab$. (See Mathematische Annalen, loc. cit., p. 89.) Likewise, if it is also externally convex, there is at least one point c for which (b) and (c) in Lemma 2 are valid.

Consequently, if we define a segment, ray, or open line as a set of points congruent with a euclidean segment, ray, or open line, respectively, we have the following results from Menger's work and Lemma 2:

THEOREM 1. *If a convex complete space Z has the four-point property, there is one and only one segment joining any pair of points a and b. This is the set $\{x\}$ for which $ab = ax + xb$.*

THEOREM 2. *If a convex complete space Z has the four-point property, the set of all points collinear with any two points is a segment, ray, or open line.*

If the two points in Theorem 2 are a and b, we call the set of collinear points the line determined by a and b. The line will be an open line if Z is both convex and externally convex; otherwise it may be bounded at one or both ends. It is an easy deduction that a line is determined by any two of its points, and that two lines can have but one point in common.

5. THEOREM 3. *Let Z be a convex complete space which has the four-point property. Let L, M, and N be three different lines in Z having one point a in common. Then $L + M + N$ can be imbedded in E_3.*

PROOF. Let b, c, and d lie on L, M, and N, respectively. By hypothesis, a, b, c, and d are congruent to points a', b', c', and d' in some E_3. This congruence defines congruences of L, M, and N, respectively, to L', M', and N', which are euclidean segments, rays, or open lines through a' and b', a' and c', and a' and d', re-

spectively. Obviously $L' \cdot M' = L' \cdot N' = M' \cdot N' = a'$. For any point x, say on L, there is a unique corresponding point x' on L' given by the congruence $L \cong L'$. To show that $L+M+N \cong L'+M'+N'$, we must prove that, if $x \sim x'$ and $y \sim y'$, then $xy = x'y'$.

To fix the ideas let x lie on L and y on M. By the hypothesis and Lemma 1, there is a point x'' such that $a+b+c+x \cong a'+b'+c'+x''$. Of the points a, b, and x, one lies between the other two, say b between a and x. Then $ax = ab+bx$ and $a'x'' = a'b'+b'x''$. Thus $a'x'' = ax$ and x'' lies on $a'b'$ produced. By the congruence $L \cong L'$, $a'x' = a'b'+b'x'$ and x' lies on $a'b'$ produced. Since $a'x' = ax = a'x''$, we have $x' = x''$. Thus $a+b+c+x \cong a'+b'+c'+x'$.

Again applying the four-point hypothesis and Lemma 1, we have a point y'' such that $a+c+x+y \cong a'+c'+x'+y''$. To fix the ideas, let y lie between a and c. Then $ay+yc = ac$ and $a'y''+y''c' = a'c'$. By the congruence $M \cong M'$, $a'y'+y'c' = a'c'$. Since $a'y' = a'y''$ and $y''c' = y'c'$, we have $y' = y''$. But $xy = x'y''$; this gives $xy = x'y'$, which was to be proved.

COROLLARY. *Under the above hypotheses two intersecting lines can be imbedded in a plane.*

6. THEOREM 4. *Let Z be a convex complete space which has the four-point property. Let a, b, and c be non-collinear points in Z which are congruent to the points a', b', and c' in some E_2. Then** $ab+bc+ca \cong a'b'+b'c'+c'a'$.

PROOF. The given congruence defines congruences $ab \cong a'b'$, $bc \cong b'c'$, and $ca \cong c'a'$; and therefore a one to one correspondence between the points $\{x\}$ of $ab+bc+ca$ and the points $\{x'\}$ of $a'b'+b'c'+c'a'$. We must prove that, if $x \sim x'$ and $y \sim y'$, then $xy = x'y'$.

To fix the ideas let x lie on ac and y on bc. By the four-point hypothesis and Lemma 1, there is a point y'' in the space containing E_2 such that $a+b+c+y \cong a'+b'+c'+y''$. Now $b'y''+y''c' = b'c'$ and $b'y'+y'c' = b'c'$. Since $b'y'' = by = b'y'$ and $y''c' = yc = y'c'$, we have $y'' = y'$.

* This congruence is a congruence between the two sets of three segments; it does not imply anything regarding the set of points lying on segments joining a to points of the segment bc and the point set forming the interior of the euclidean triangle $a'b'c'$.

Repeating this reasoning, we find that there is an x'' such that $a+c+y+x \cong a'+c'+y'+x''$. Then $a'x''+x''c'=ax+xc=ac=a'c'$ and $a'x'+x'c'=ax+xc=ac=a'c'$. Hence $x'=x''$. Therefore $xy=x''y'=x'y'$, which was to be proved.

7. *Angle between two Lines.* In consequence of the two preceding sections the angle between two lines in a convex complete space having the four-point property can be identified with the euclidean angle between their congruent images. Thus, if a, b, and c are the vertices of a "triangle," the cosine of the angle bac is

$$\frac{(ab)^2+(ac)^2-(bc)^2}{2(ab)(ac)}.$$

With regard to such questions as tangency to a simple arc we are handicapped by the absence of a coördinate system and the fact that it may be impossible to imbed any more than four points of the arc in any one euclidean space. There is, however, no difficulty in defining tangents.

Let C be a simple arc with its points ordered from left to right and let a be one of its points. Let L be a half-line having a as one end and passing through a point x at the right (left) of a on C, and let T be a half-line from a passing through some point t. If the angle xat approaches 0 as $x \to a$, we say that T is a right-hand (left-hand) tangent to C at a. In consequence of §5 and the properties of euclidean space angles obey the rule that angle bac+angle $bad \geqq$ angle cad; hence there cannot be more than one right-hand or left-hand tangent at a point.

If S is a left-hand and T a right-hand tangent at a, there are three possibilities: a cusp, an angle-point, or an ordinary tangent. The first occurs when the angle between S and T is zero, or $S=T$; the last occurs when S and T make an angle of π, or $S+T$ is a line; and we have an angle-point when the angle between S and T is neither 0 nor π.

To obtain necessary and sufficient conditions for the existence of tangents we need the following lemma from plane trigonometry.

8. LEMMA 3. *Let axy be a triangle congruent with a plane triangle and $ax \leqq ay$. In order that angle $xay \to 0$ as $ax+ay \to 0$ it is necessary and sufficient that $(ax+xy-ay)/(ax) \to 0$; in order that*

angle $xay \to \pi$ *as* $ax+ay \to 0$ *it is necessary and sufficient that* $(ax+ay-xy)/(ax) \to 0$.

PROOF. Let θ be the angle xay. Then by the law of cosines

$$1 - \cos\theta = \frac{(ax + xy - ay)(ay + xy - ax)}{2(ax)(ay)}. \tag{1}$$

Now $\theta \to 0$ if and only if $1-\cos\theta \to 0$. Since $|xy-ax| \leqq ay$, it is clear that $1-\cos\theta \leqq (ax+xy-ay)/(ax)$; hence the condition is sufficient.

To prove the converse we set $(ax+xy-ay)/(ax) = m$. Then m is a non-negative function of the sides of the triangle, and

$$ax + xy - ay = m \cdot ax. \tag{2}$$

Now $2(ay-ax) \geqq 0$. Adding this to (2), we have $ay+xy-ax \geqq m \cdot ax$, and (1) gives

$$1 - \cos\theta \geqq \frac{m^2 \cdot ax}{2 \cdot ay}. \tag{3}$$

Since $1-\cos\theta \to 0$, either $m \to 0$ or there is a sequence of triangles $\{ax_iy_i\}$ for which $ax_i \leqq ay_i$, $ax_i \to 0$, and $ay_i \to 0$ as $i \to \infty$, and m has a positive lower bound k. Then $ax_i/ay_i \to 0$ and for i large enough the relation $ax_i + x_iy_i \geqq ay_i$ gives $x_iy_i \geqq ax_i$, or

$$\frac{ay_i + x_iy_i - ax_i}{2 \cdot ay_i} \geqq \frac{1}{2}.$$

But then (1) and (2) give

$$1 - \cos\theta \geqq k/2,$$

which is a contradiction. Hence the first part of the theorem is proved.

For the second part we must examine $1+\cos\theta$, which approaches 0 as $\theta \to \pi$. Now

$$1 + \cos\theta = \frac{(ax + ay + xy)(ax + ay - xy)}{2(ay)(ax)}.$$

Since $xy \leqq ax+ay$, $ax+ay+xy \leqq 2(ax+ay) \leqq 4(ay)$. Thus we have $1+\cos\theta \to 0$ if $(ax+ay-xy)/(ax) \to 0$. On the other hand, we have $(ax+ay+xy)/[2(ay)] > 1/2$; hence the condition is necessary.

9. THEOREM 5. *Let C be a simple arc in a convex complete space Z having the four-point property. Let C have a right-hand (left-hand) tangent at a point a and let x and y be points of C at the right (left) of a with $ax \leqq ay$. Then, as x and y both approach a, $(ax+xy-ay)/(ax) \to 0$.*

PROOF. Let T be the tangent and t be a point of T different from a. By the definition of a unilateral tangent, angle $tax \to 0$ as $x \to a$ and angle $tay \to 0$ as $y \to a$.

Since angle $xay \leqq$ angle $tax +$ angle tay, angle $xay \to 0$ as x and y both approach a. Then Lemma 3 gives the theorem:

10. THEOREM 6. *Let C be a simple arc in a convex complete space Z having the four-point property. At a point a let there be a tangent and let x and y be points of C on opposite sides of a, with $ax \leqq ay$. Then, as x and y both approach a, $(ax+ay-xy)/(ax) \to 0$.*

PROOF. Let the tangent T at a be the union of the unilateral tangents T' and T'' containing points s and t, respectively, distinct from a. Let x be at the left of a and T' be the left-hand tangent. Then angle $xas \to 0$ as $x \to a$ and angle $yat \to 0$ as $y \to a$. By §5 the tangent T and the lines determined by a and x, and by a and y can be imbedded in E_3. Consequently angle $xay \to \pi$ as x and y both approach a. The desired conclusion now follows from §8.

11. THEOREM 7. *Let Z be a complete metric space which has the four-point property and is both convex and externally convex. Let C be a simple arc and a be any point not an end-point. Then the conditions of §§9 and 10 are sufficient for the existence of unilateral and ordinary tangents, respectively.*

PROOF. Let us take the case of right-hand tangents. Let x and y be any points at the right of a and C, $ax \leqq ay$, and $\theta(x, y)$ be the angle xay. By virtue of the hypothesis, the continuity of the cosine function, and §8, we have for any positive ϵ a positive δ such that

(1) $\qquad \theta(x, y) < \epsilon/2$ if $ax < \delta$ and $ay < \delta$.

Let $\{x_i\}$ be a sequence of points of C at the right of a and $x_i \to a$ as $i \to \infty$. For every i greater than some i', $ax_i < \delta$. Then (1) becomes

(2) $$\theta(x_i, x_j) < \epsilon/2 \text{ if } i \geqq i' \text{ and } j \geqq i'.$$

Since Z is externally convex there is a point r_i on each half-line from a through x_i such that $ar_i = 1$. By (2) and elementary trigonometry the sequence of points $\{r_i\}$ is a Cauchy sequence and converges to a limit r since Z is complete. Let T be the half-line from a through r.

Since $r_i \to r$, we have for any positive ϵ an i'' for which

(3) $$\theta(r, x_i) < \epsilon/2 \text{ if } i \geqq i''.$$

Relations (1) and (3) now give $\theta(r, x) < \epsilon$ if $ax < \delta$. Hence T is a right-hand tangent by definition.

Let us now consider the case of the ordinary tangent. In consequence of the first part of the proof we know that, if there is no right-hand tangent, there is a constant $k > 0$ and two sequences $\{y_i\}$ and $\{y_i'\}$ of points of C at the right of a such that $y_i \to a$, $y_i' \to a$, and

(4) $$\theta(y_i, y_i') > k.$$

From the hypothesis and §8 we know that for any positive ϵ, any point x on C at the left of a and sufficiently near to a, and i greater than some i',

(5) $$\pi - \theta(x, y_i) < \epsilon;$$

(6) $$\pi - \theta(x, y_i') < \epsilon.$$

Taking $\epsilon < k/4$, we have by (4), (5), and (6)

(7) $$\theta(x, y_i) + \theta(x, y_i') + \theta(y_i, y_i') > 2\pi + k/2.$$

Since the lines through a and x, y_i, y_i', respectively, can be imbedded in E_3, relation (7) is false.

Thus there is a right-hand and likewise a left-hand tangent. These form a line since $\theta(x, y) \to \pi$ as $x \to a$ and $y \to a$.

The statement and proof of the theorem for the special cases where a is an end-point are obvious.

12. *Conclusion.* It will be noted that the convergence to zero of $(ax+xy-ay)/(ax)$ when x and y are on the same side of a or of $(ax+ay-xy)/(ax)$ when x and y are on opposite sides of a as x and y approach a on the arc C with $ax \leqq ay$ is an *intrinsic*

property of the arc and may be true of a particular simple arc in any metric space. In the first case we may say that C is *smooth at the right (left)* of a, in the second that C is *smooth about* a. The property of smoothness is allied on the one hand to the triangle axiom of metric geometry and on the other to the well known fact from elementary geometry that the distance from a point x on a curve to a tangent at a is an infinitesimal of the second order with respect to ax as $x \to a$.

For the existence of a tangent line the condition of external convexity or something like it is needed because, as $x \to a$, the half-line from a through x may shrink to a point, as would be the case if our curve C were a circle and our space Z the union of C and its interior.

In the absence of the four-point property we have the somewhat surprising result that smoothness about a point needs not imply unilateral smoothness. To see this, let us take the following points in the plane: $a=(0,0)$, $b=(-1,0)$, $e=(1,0)$, $f=(1,1)$, $e_i=(1/2^i, 0)$, $f_i=(1/2^i, 1/2^i)$, $i=1,2, \cdots$. Let Z be the union of the segments be, ef, af, ef_1, $\{e_if_i\}$, and $\{e_if_{i+1}\}$, $i=1, 2, \cdots$; and let the distance between any two points of Z be the length of the shortest broken line joining them. Then Z is metric, compact, and convex. Now the union of the set $Z-(af+ae)$ and its limiting points is a simple arc C joining b and f. If x and y are points on the opposite sides of a, the shortest broken line from x to y passes through a; hence $xy=ax+ay$ and so C is smooth about a. On the other hand, if x and y are successively the points $\{e_i\}$ and $\{f_i\}$, respectively,

$$\frac{ax + xy - ay}{ax} = \frac{ae_i + e_if_i - af_i}{ae_i} = \frac{ae_i(2 - \sqrt{2})}{ae_i} = 2 - \sqrt{2},$$

and so C is not smooth on the right of a.

YALE UNIVERSITY

RELATION OF WHITEHEAD AND RUSSELL'S THEORY OF DEDUCTION TO THE BOOLEAN LOGIC OF PROPOSITIONS*

BY B. A. BERNSTEIN

1. *Introduction.* Whitehead and Russell's theory of deduction for "elementary" propositions, which underlies their *Principia Mathematica*, is "the calculus of propositions"; it is "the theory of how one proposition can be inferred from another"; the primitives of the theory "are sufficient for all common forms of inference." The theory thus seems to be designed to yield the same facts of logic as are given by the classic Boolean logic of propositions. But our authors tell us that "the theory of classes and the theory of propositions · · · are not coordinate," that "the theory of propositions necessarily precedes the theory of classes." The theory of deduction then seems to be essentially different from the Boolean logic of propositions. The latter logic is coordinate with the logic of classes; it in no way precedes the logic of classes. Just what is the mathematical relation of the theory of deduction to the Boolean logic of propositions? The object of my paper is to answer this question.

2. *Boolean Form of the Theory of Deduction.* A mathematical comparison between the theory of deduction and the Boolean logic of propositions requires that the two logics be expressed in a common mathematical language. This common language I have provided in a previous paper†, when I rewrote the theory of deduction in the language of Boolean algebra. I begin my discussion by presenting the essentials of the two logics in Boolean form.

In Boolean form, the theory of deduction is a system $(K, ', +)$, consisting of an undefined *class* K, an undefined *unary operation* "$'$", and an undefined *binary operation* "$+$", satisfying the following eight postulates:

---

* Presented to the Society, June 20, 1929, as part of a paper entitled *On Whitehead and Russell's theory of deduction.*

† *Whitehead and Russell's theory of deduction as a mathematical science*, this Bulletin, vol. 37 (1931), pp. 480–488.

1.1. There exists a K-element 1 such that from $a=1$ and $a'+b=1$ follows $b=1$.

1.2. $(a+a)'+a=1$.

1.3. $b'+(a+b)=1$.

1.4. $(a+b)'+(b+a)=1$.

1.5. $[a+(b+c)]'+[b+(a+c)]=1$.

1.6. $(b'+c)'+[(a+b)'+(a+c)]=1$.

1.7. If a is a K-element, a' is a K-element.

1.71. If a and b are K-elements, $a+b$ is a K-element.

The important defined ideas in the theory are $a \supset b$, $a \cdot b$ (or ab), $a \equiv b$. The definitions are

DEF. 1.01. $(a \supset b)=a'+b$,

DEF. 3.01. $ab=(a'+b')'$,

DEF. 4.01. $(a \equiv b)=(a \supset b)(b \supset a)=(a'+b)(b'+a)$.

3. *The Boolean Logic of Propositions.* The Boolean logic of propositions may be characterized as a system $(K, +)$, consisting of an undefined *class* K and an undefined *binary operation* "$+$", satisfying the following three postulates*:

A. There is a K-element 0 such that $0+a=a$ for every K-element a.

B. There is a K-element 1 such that $1+a=1$ for every K-element a.

C. K consists of two elements.

The important defined ideas of this system are a', $a+b$ (or ab), $a<b$. The definitions are:

DEF. 1. $0'=1$, $1'=0$,

DEF. 2. $ab=(a'+b')'$,

DEF. 3. $(a<b)=(a'+b=1)$.†

* See my *Sets of postulates for the logic of propositions*, Transactions of this Society, vol. 28 (1926), pp. 472–478.

† Note that $a<b$ is not the same as $a \supset b$ of §2. As a concrete science, $(K, +)$ is an algebra of "truth-values," *truth* 1 and falsity 0. Let p, q be two propositions and a, b their respective truth-values. Then a', $a+b$, ab are the respective truth-valves of the propositions "not $-p$," "p or q," "p and q"; the relation $a<b$ signifies "p implies q," and $a=b$ signifies "p is equivalent to q."

The operations $a+b$ and ab and the relation $a<b$, as determined by postulates A-C and definitions 2, 3, are given by the following tables:*

(i)

$+$	0	1
0	0	1,
1	1	1

(ii)

$\times$	0	1
0	0	0,
1	0	1

(iii)

$<$	0	1
0	+	+.
1	−	+

Each of these tables shows that *the Boolean logic of propositions is a two-element logic of classes.*

4. *The Theory of Deduction Derivable from the Boolean Logic of Propositions.* Now that we have before us the theory of deduction and the Boolean logic of propositions expressed in a common language, I can proceed to answer the question, raised in § 1, concerning the relation between the two theories. I observe, first, that *the theory of deduction is derivable from the Boolean logic of propositions.*

This fact can easily be verified by deriving propositions 1.1 –1.71 of §2 from propositions A–C of §3, or by using table (i) of §3. As illustration of the use of table (i), I give the proofs of 1.1 and 1.6, perhaps the least simple of the proofs.

PROOF of 1.1. The element 1 in postulate B is the element 1 required in 1.1. For suppose $a=1$ and $a'+b=1$, that is, suppose $1'+b=1$. Then $b=1'+b=1$, by table (i), def. 1, hyp.

PROOF OF 1.6. (1) $a'+a=1$, by def. 1 and table (i).

Further, for $a=0$, the left member of 1.6 is

$$(2)\quad (b'+c)'+[(0+b)'+(0+c)]=(b'+c)'+(b'+c)=1,$$

by table (i) and (1).

For $a=1$, the left member of 1.6 is

$$(3)\quad (b'+c)'+[(1+b)'+(1+c)]=(b'+c)'+(1'+1)$$
$$=(b'+c)+1=1,$$

by table (i), (1), table (i).

Hence the theorem follows, by (2) and (3).

---

* In reading the tables, take a from the extreme left column and b from the uppermost row. In table (iii) a "+" sign denotes that $a<b$ holds, a "−" sign that $a<b$ does not hold.

5. *The Boolean Logic of Propositions Not Derivable from the Theory of Deduction.* I observe, next, that *the Boolean logic of propositions is not derivable from the theory of deduction.*

For, the following system $(K, ', +)$ is an independence-system for proposition C with respect to 1.1–1.71:

K, a class consisting of the single element e; $e'=e$; $e+e=e$.

Of course, all propositions in the Boolean logic of propositions that are dependent on C are propositions not derivable from the theory of deduction. Among such propositions, for example, are the following:

D. For every K-element a, either $a=1$ or else $a=0$.

E. $a'\neq a$.

F. If $a<b$ or $a<c$ then $a<b+c$, and conversely.

That D-F are propositions in the Boolean logic of propositions can easily be verified with the help of table (i).

That D is not derivable from 1.1–1.71 is seen from the following independence-system $(K, ', +)$:

K, the infinite set of plane regions contained in a plane region T, including T and including the null region N; a', the region in T outside a; $a+b$, the smallest region containing a and b.

An independence-system for E with respect to 1.1–1.71 is the independence-system for C above.

An independence-system for F is the system used in proving the independence of D.*

6. *Inadequacy of the Theory of Deduction as "the Calculus of Propositions."* The fact that propositions D-F are unobtainable from 1.1–1.71 shows that the theory of deduction is inadequate as "the calculus of propositions." The theory of deduction is "the theory of how one proposition can be inferred from another"; its primitives "are sufficient for all common forms of inference"; it is concerned with "the principles by which conclusions are inferred from premisses"; it is designed to yield "the rules" from which "the usual calculus of formal logic starts." The theory, as "the calculus of propositions," bears the same name as Schröder's *Der Aussagenkalkül.* But the theory of de-

* Compare my paper *On proposition* *4.78 *of Principia Mathematica*, this Bulletin, vol. 38 (1932), pp. 388–391.

duction cannot yield such propositions as D-F above, propositions which are fundamental in the classic theory† and which the theory of deduction aims to obtain.‡ Clearly, the theory of deduction is inadequate as "the calculus of propositions."

7. *Conclusion.* I have answered completely the question of the relation of the theory of deduction of the *Principia* to the Boolean logic of propositions. *The theory of deduction is derivable from the Boolean logic of propositions, but not conversely.* The latter fact makes the theory inadequate as "the calculus of propositions."

THE UNIVERSITY OF CALIFORNIA

---

† Proposition D is of particular importance in the classic theory. For D is essentially the same as proposition C above, and C, as I have shown elsewhere, *can be made to serve as a one-postulate set for the Boolean logic of propositions.* (See my *Sets of postulates for the logic of propositions*, cited above.)

‡ The authors of the *Principia* no doubt regard D-F as equivalent to their

* 2.11. $\vdash \cdot p \vee \sim p$,

* 5.19. $\vdash \sim(p \equiv \sim p)$,

* 4.78. $\vdash \cdot [(p \supset q) \vee (p \supset r)] \equiv [p \supset (q \vee r)]$.

For they would interpret these propositions respectively as:

D_0. Every proposition is true or else false,

E_0. A proposition is never equivalent to its contradictory,

F_0. If p implies q, or p implies r, then p implies "q or r"; and conversely. Interpretations D_0-F_0 are, however, all inadmissible. They are based on reading the fundamental symbols $\sim p$, $p \vee q$, $p \supset q$, $p \cdot q$, $p \equiv q$ respectively as "p is false," "p is true or q is true," "p implies q", "p is true and q is true," "p is equivalent to q"—readings which all violate the distinction between "p" and "$\vdash \cdot p$" made by the authors earlier in their theory. The correct readings of the fundamental symbols should be, respectively, "not $-p$," "p or q," "not $-p$ or q," "p and q," "'not $-p$ or q' and 'not $-q$ or p'". (Compare the remarks on "$\vdash$" and "$\supset$" in my review of Volume I of the *Principia*, this Bulletin, vol. 32, pp. 711–713. With regard to *4.78, compare also my paper on *4.78, cited above.)

ON ALGEBRAIC EQUATIONS HAVING ONLY REAL ROOTS*

BY W. E. ROTH

Given the algebraic equation

$$(1)\quad f_1(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0,\ a_n \neq 0,$$

whose roots x_i, $i=1, 2, \cdots, n$, are all real. Then there exist equations, $f_\lambda(x)=0$, also of degree n, whose roots are the real numbers x_i^λ, $(i=1, 2, \cdots, n; \lambda=\pm 1, \pm 2, \cdots)$. If

$$(2)\quad S_j = x_1^j + x_2^j + \cdots + x_n^j, \qquad (j = 0, \pm 1, \pm 2, \cdots),$$

then the determinants

$$\Delta_k^{(\lambda)} = \begin{vmatrix} S_0 & S_\lambda & \cdots & S_{(k-1)\lambda} \\ S_\lambda & S_{2\lambda} & \cdots & S_{k\lambda} \\ \cdots & \cdots & \cdots & \cdots \\ S_{(k-1)\lambda} & S_{k\lambda} & \cdots & S_{(2k-2)\lambda} \end{vmatrix}, \qquad (k = 2, 3, \cdots, n),$$

are all positive according to Borchardt's theorem† provided the roots of $f_\lambda(x)=0$ are distinct. On the other hand, if $f_1(x)=0$ has exactly μ $(\mu \leqq n)$ distinct roots, all real, then for odd values of λ, $f_\lambda(x)=0$ will have exactly the same number of distinct roots; and for even values of λ, exactly $\mu-\nu$ distinct roots, where ν is the number of distinct pairs of numerically equal roots of $f_1(x)=0$ which differ only in sign. Under these hypotheses, it is known that

$$\Delta_k^{(\lambda)} > 0, \qquad (k = 2, 3, \cdots, \mu),$$
$$= 0, \qquad (k = \mu+1, \mu+2, \cdots, n),$$

if λ is an odd positive or negative integer, and that

$$\Delta_k^{(\lambda)} > 0, \qquad (k = 2, 3, \cdots, \mu-\nu),$$
$$= 0, \quad (k = \mu-\nu+1, \mu-\nu+2, \cdots, n),$$

* Presented to the Society, November 28, 1931.

† Borchardt, Journal de Mathématiques, vol. 12 (1847), p. 58; Werke, Berlin, (1888), p. 24.

if λ is an even positive or negative integer.* Upon the basis of the above known results we shall establish the following theorem and shall develop some of its implications.

THEOREM. *If the roots x_i, $(i=1, 2, \cdots, n)$, of the equation*

$$f_1(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0, \quad a_n \neq 0,$$

are all real and if we use the notation (2), *then all roots of $f_1(x)=0$, whose multiplicity equals or exceeds r, must lie on the real intervals where the following inequalities in x hold*:

$$\Delta_{k,r}^{(\lambda)}(x) \equiv \begin{vmatrix} S_0 - r & S_\lambda - rx^\lambda & \cdots & S_{(k-1)\lambda} - rx^{(k-1)\lambda} \\ S_\lambda - rx^\lambda & S_{2\lambda} - rx^{2\lambda} & \cdots & S_{k\lambda} - rx^{k\lambda} \\ \cdots & \cdots & \cdots & \cdots \\ S_{(k-1)\lambda} - rx^{(k-1)\lambda} & S_{k\lambda} - rx^{k\lambda} & \cdots & S_{(2k-2)\lambda} - rx^{(2k-2)\lambda} \end{vmatrix} \geqq 0,$$

$$(k=2, 3, \cdots, n-r+2).$$

Let x_i be any root of $f_1(x)=0$ whose multiplicity equals or exceeds r, then there exist equations

$$F_{i,r}^{(\lambda)}(x) \equiv \frac{f_\lambda(x)}{(x - x_i^\lambda)^r} = 0, \qquad (\lambda = \pm 1, \pm 2, \cdots), \tag{3}$$

for $i=1, 2, \cdots, \mu$, of degree $n-r$ in x, whose roots are identical with those of $f_\lambda(x)=0$ save that the multiplicity of the root, x^λ, is reduced by r. The sum of the jth powers of the roots of (3) is then given by $S_{j\lambda} - rx_i^{j\lambda}$, where $S_{j\lambda}$ is given by (2). Now the roots of $F_{i,r}(x)=0$ are all real, hence the first principal minors of its discriminant are $\Delta_{k,r}^{(\lambda)}(x_i)$, $(k=2, 3, \cdots, n-r)$, and all these minors, where r does not exceed the multiplicity of the root, x_i, of $f_1(x)=0$, are positive or zero. If $f_\lambda(x)=0$ has μ distinct roots x_i^λ, $(i=1, 2, \cdots, \mu)$, of multiplicity r_i, respectively, then we can readily show that

$$\Delta_{\mu+1,r}^{(\lambda)}(x) \equiv -\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x^\lambda & x_1^\lambda & \cdots & x_\mu^\lambda \\ \cdots & \cdots & \cdots & \cdots \\ x^{\mu\lambda} & x_1^{\mu\lambda} & \cdots & x_\mu^{\mu\lambda} \end{vmatrix}^2 r\prod_{i=1}^{\mu} r_i. \tag{4}$$

* Baur, Mathematische Annalen, vol. 50 (1898), p. 241. Grommer, Journal für Mathematik, vol. 144 (1914), p. 114.

Hence $\Delta^{(\lambda)}_{\mu+1,r}(x)=0$ only for values of x such that $f_1(x)=0$ and is negative for all other values of x, whatever r may be. Thus the equality sign given in the theorem above holds in this particular case. If $\mu=n$ we find, as a direct consequence of (4), that

$$\Delta^{(\lambda)}_{n+1,1}(x) \equiv f_1^2(x)K(x),$$

where $K(x)$ is a polynomial in x. For $k \geqq \mu+2$

$$\Delta^{(\lambda)}_{k,r}(x) \equiv 0,$$

as can be readily verified by expanding this determinant in powers of x^λ. All coefficients of the expansion are the sums of minors of order $k-1$ of $\Delta^{(\lambda)}_{k,0}(x)$, and are consequently zeros. That is, if k exceeds by two or more the number of distinct roots of $f_\lambda(x)=0$, $\Delta^{(\lambda)}_{k,r}(x)$ is identically zero and the theorem is satisfied. An additional special case of interest is the following. If $\mu=n$ or $n-1$,

$$\Delta^{(\lambda)}_{n,1}(x) \equiv f_1(x)L(x),$$

where $L(x)$ is a polynomial in x. If $\mu=n-1$, this identity holds because of (4). Therefore we shall assume that $\mu=n$. Evidently

$$\Delta^{(\lambda)}_{n,1}(x_i) = 0, \qquad (i = 1, 2, \cdots, n);$$

since this is a determinant of order n in the elementary symmetric functions of the $n-1$ roots of the equation $F^{(\lambda)}_{i,r}(x)=0$, $(i=1, 2, \cdots, n)$, respectively, and the identity above is proved. Hence we have shown that $\Delta^{(\lambda)}_{k,r}(x)$ must be positive or zero on any interval which contains a root of $f_1(x)=0$ whose multiplicity equals or exceeds r.

We shall now prove a generalization of a theorem due to Laguerre* as a corollary of the above theorem.

COROLLARY 1. *If $f_1(x)=0$ has the real roots x_i, $(i=1, 2, \cdots, n)$, not necessarily all distinct, but $f(0)\neq 0$, and if we use the notation (2), then all roots of $f_1(x)=0$ whose multiplicity equals or exceeds r, must lie on the interval where the inequalities*

* Laguerre, *Sur une méthode pour obtenir par approximation les racines d'une équation algébrique qui a toutes ses racines réelles*, Nouvelles Annales de Mathématiques, (2), vol. 19 (1880), pp. 161–171; see particularly the footnote on page 169. See also Kraus, Časopis pro Pěstování Mathematiky a Fysiky, vol. 15 (1886), p. 63.

$$(5)\qquad \frac{rS_\lambda - [r(n-r)(nS_{2\lambda} - S_\lambda^2)]^{1/2}}{nr} \leqq x^\lambda \leqq \frac{rS_\lambda + [r(n-r)(nS_{2\lambda} - S_\lambda^2)]^{1/2}}{nr},$$

$(\lambda = \pm 1, \pm 2, \cdots)$, *hold for real* x.

This corollary follows directly from the inequality $\Delta_{2,r}^{(\lambda)}(x) \geqq 0$. In this as in (5) the equality signs hold only if $f_\lambda(x) = 0$ has an r-fold root and at most one s-fold root distinct from it. The radical in (5) is always real because $nS_{2\lambda} - S_\lambda^2$ is the first principal minor of order two of the discriminant of $f_\lambda(x) = 0$, and is positive. The limits between which all roots whose multiplicity equals or exceeds r must lie, according to the corollary above, are expressed in terms of the 2λ coefficients $a_1, a_2, \cdots, a_{2\lambda}$ of $f_1(x)$ in case λ is a positive integer and in terms of the $1-2\lambda$ coefficients, $a_{n+2\lambda}, a_{n+2\lambda-1}, \cdots, a_n$, in case λ is a negative integer. We shall now show that the intervals where roots of $f_1(x) = 0$ must lie are reduced as λ is increased in numerical value. If α is the absolute value of the numerically greatest root of $f_1(x) = 0$, then

$$\frac{S_\lambda}{\alpha^\lambda} = \left(\frac{x_1}{\alpha}\right)^\lambda + \left(\frac{x_2}{\alpha}\right)^\lambda + \cdots + \left(\frac{x_n}{\alpha}\right)^\lambda = \tau_1 + a, \qquad (|\tau_1| < n),$$

where τ_1 is a positive or negative integer giving the sum of the terms in which $|x_i| = \alpha$, and a is the sum of the remaining terms of the right member in which $|x_i| < \alpha$. In the following we shall assume that $\tau_1 > 0$. Similarly

$$S_{2\lambda}/\alpha_{2\lambda} = \tau_2 + b > 0, \qquad (n > \tau_2 > 0),$$

where τ_2 is the sum of the terms in which $|x_i| = \alpha$ and b is the sum of those in which $|x_i| < \alpha$. If λ is even, $\tau_1 = \tau_2$. By taking λ sufficiently large a and b can both be made as small as we please for each $(x_i/\alpha)^\lambda$, $(x_i < \alpha)$, approaches zero as λ grows. From (5) we get

$$\left(\frac{x}{\alpha}\right)^\lambda \leqq \frac{\tau_1 + a}{n} + \left\{\frac{n-r}{r}\left[\frac{\tau_2 + b}{n} - \left(\frac{\tau_1 + a}{n}\right)^2\right]\right\}^{1/2}.$$

Now $nS_{2\lambda} \geqq S_\lambda^2$; hence we have

$$\frac{\tau_2 + b}{n} \geqq \left(\frac{\tau_1 + a}{n}\right)^2.$$

But for positive λ, $(\tau_2+b)/n<1$; hence

$$1 > \frac{\tau_2 + b}{n} \geqq \left(\frac{\tau_1 + a}{n}\right)^2,$$

or

$$1 > 1 - \left(\frac{\tau_1 + a}{n}\right)^2 > \frac{\tau_2 + b}{n} - \left(\frac{\tau_1 + a}{n}\right)^2 \geqq 0;$$

that is $(\tau_1+a)/n<1$, if $\tau_1>0$. Then

$$\left(\frac{x}{\alpha}\right)^\lambda \leqq 1 + \left(\frac{n-r}{r}\right)^{1/2} \leqq 1 + (n-1)^{1/2},$$

and

$$x \leqq (1 + (n-1)^{1/2})^{1/\lambda}\alpha.$$

The coefficient of α approaches unity as λ increases. Similarly if $\tau_1<0$ and λ is an odd positive integer,

$$x \geqq -[1 + (n-1)^{1/2}]^{1/\lambda}\alpha.$$

Hence the numerically greater bound obtained from (5) when λ is positive approaches the root of greatest numerical value as λ is given larger and larger integral values. Similarly the numerically lesser bound given by (5) for negative values of λ approaches the root of least numerical value as $-\lambda$ is given ever larger integral values.

The theorem of Laguerre mentioned above is obtained from our corollary by letting $\lambda=1$ and $r=1$. Laguerre gave two proofs of his theorem either of which, by the use of the polynomial $f_\lambda(x)$, could have been extended to cover the case $\lambda>1$. This obvious extension seems to have escaped later writers who proved the theorem. A generalization of the theorem was made by Pleskot;* its proof rests upon the inequality

* Pleskot, *Über die Grenzen der Wurzeln einer Gleichung mit nur reellen Wurzeln*, Sitzungsberichte der Böhmischen Gesellschaft der Wissenschaften in Prag, 1897, No. 37; *Limites des racines d'une équation n'ayant que des racines réelles*, Nouvelles Annales de Mathématiques, (3), vol. 18 (1899), pp. 301–305.

$$\frac{S_{2\lambda} - x^{2\lambda}}{n-1} \geqq \left(\frac{S_1 - x}{n-1}\right)^{2\lambda}, \quad (\lambda = +1, 2, \cdots),$$

which he proved. However, since $\Delta_{2,1}^{(2p)}(x) \geqq 0$, we have at once the inequality

$$\frac{S_{4p} - x^{4p}}{n-1} \geqq \left(\frac{S_{2p} - x^{2p}}{n-1}\right)^2,$$

which is only another way of writing (5) when $\lambda = 2p$. From this and Pleskot's inequality we have

$$\frac{S_{4p} - x^{4p}}{n-1} \geqq \left(\frac{S_{2p} - x^{2p}}{n-1}\right)^2 \geqq \left(\frac{S_1 - x}{n-1}\right)^{4p}.$$

Pleskot would use the first and last members of this inequality to compute his bounds and above in (5) we use the first two members, hence for even values of λ we get at least as good bounds for the roots as does Pleskot. If λ is an odd integer the same conclusion may be valid but has not been proved by the writer. Take the equation

$$f_1(x) = (x+4)(x+1)(x-2)(x-3) = 0$$

as an example to illustrate the latter case. Pleskot's inequality

$$\frac{S_6 - x^6}{n-1} \geqq \left(\frac{S_1 - x}{n-1}\right)^6$$

gives us the limits

$$-4.117 \leqq x \leqq 4.117,$$

and

$$\frac{S_6 - x^6}{n-1} \geqq \left(\frac{S_3 - x^3}{n-1}\right)^2,$$

which is (5), for $\lambda = 3$, $r = 1$, gives us $-4.055 \leqq x \leqq 3.725$. Both limits are nearer the extreme roots of the given equation in the latter than in the former inequality.

Nagy* gives another demonstration of Laguerre's theorem, that is, of Corollary 1 for the case $\lambda = 1$, $r = 1$, which he bases upon a theorem by Schur.

---

* Nagy, *Über algebraische Gleichungen mit lauter reellen Wurzeln*, Jahresbericht der deutschen Mathematiker Vereinigung, vol. 27 (1918), pp. 37–45.

COROLLARY 2. *If λ is an odd integer and if $S_\lambda^2 > (n-r)S_{2\lambda}$, then all roots of $f_1(x)=0$ whose multiplicity equals or exceeds r are positive or all are negative according as S_λ is positive or negative.*

This proposition follows at once from Corollary 1. When it is satisfied the relationship (5) provides upper and lower bounds for the roots in question whether λ be positive or negative. However, if $f_1(x)=0$ has both positive and negative r-fold roots, then $S_\lambda^2 < (n-r)S_{2\lambda}$, where λ is an odd positive or negative integer; and in this case (5) provides us with an upper bound for the negative and a lower bound for the positive r-fold roots if λ is negative and extreme bounds for all r-fold roots if λ is positive.

It is easily shown that all principal minors of $\Delta_{n,r}^{(\lambda)}(x)$ are positive or zero on intervals containing r-fold roots of $f_1(x)=0$. Hence $S_{2\lambda} - rx^{2\lambda} > 0$, $(\lambda = \pm 1, \pm 2, \cdots)$, on any interval where r-fold roots of $f_1(x)=0$ lie.

A great variety of formulas bounding the roots of $f_1(x)=0$ can be based on other principal minors of $\Delta_{n,r}^{(\lambda)}(x)$ than those employed here and in Corollary 1, but they require the solution of equations of degrees higher than the second in x^λ, and are therefore not readily solved for the limiting values.

Finally we can show that

$$\frac{S_{\pm(2\lambda+2)}}{S_{\pm 2\lambda}} < x^{\pm 2},$$

where λ is positive and the subscripts and exponents are either all positive or all negative. This inequality gives a lower bound of the numerical value of the largest numerical root if the positive signs are taken, and an upper bound to the numerical value of the least numerical root if the negative signs are taken. The relation is a special case under the Lagrange-Bernoulli theorem.*

UNIVERSITY OF WISCONSIN,
EXTENSION DIVISION

* Pascal's *Repertorium*, vol. I, 1, 1910, p. 355.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

On account of the fact that the Summer Meeting of the Society in 1932 will be held on the Pacific Coast, the meetings that were to have been held at Seattle (June, 1932) and at Los Angeles (November, 1932) will be omitted.

LOS ANGELES, CALIFORNIA, SUMMER MEETING AND COLLOQUIUM, August 30-September 2, 1932.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wis., not later than July 26. Abstracts received by July 7 will appear in the July issue, or an earlier issue, of this BULLETIN. Professor J. F. Ritt will deliver a series of colloquium lectures on *Differential equations from the algebraic standpoint.* By invitation of the program committee, Professor D. N. Lehmer will speak on *The continued fraction representing cubic and higher irrationalities,* and Professor Tibor Radó will speak on *Recent work in the problem of Plateau.*

NEW YORK CITY, October 29, 1932.

Abstracts must be in the hands of Associate Secretary Tomlinson Fort, 501 West 116th St., New York City, not later than October 8. In order to be printed in the BULLETIN in advance of the meeting, abstracts must be in the hands of the Associate Secretary not later than September 7. By invitation of the program committee, Professor Jesse Douglas will speak on *The problem of Plateau.*

AMES, IOWA, November 25-26, 1932.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 3. Abstracts received by that date will appear in the November issue of this BULLETIN. By invitation of the program committee, Professor E. B. Stouffer will speak on *Some canonical forms and their associated geometries,* and Professor H. W. March will speak on *Applications of the theory of elasticity to wood, a material of anisotropic structure.*

ATLANTIC CITY, NEW JERSEY, ANNUAL MEETING, December 27-30, 1932.

Abstracts are due by November 30, 1932.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, October, 1930, p. 58), are payable to the Treasurer of the Society, Professor G. W. Mullins, 501 West 116th St., New York City.

WHOLE NUMBER 395

CONTENTS

	PAGE
Summation of Fourier Series. By EINAR HILLE	505
Note on the Law of Biquadratic Reciprocity. By V. W. ADKISSON	529
A Plane, Arcwise Connected and Connected im Kleinen Point Set which is not Strongly Connected im Kleinen. By R. L. WILDER	531
Some Involutorial Line Transformations interpreted as Points of V_2 of S_5. By J. M. CLARKSON	533
Note on a Theorem due to Bromwich. By H. L. GARABEDIAN	541
A Theorem on Symmetric Determinants. By W. V. PARKER	545
Quadratic Partitions: Paper II. By E. T. BELL	551
The Transformation of Lines of Space by means of two Quadratic Reguli. By A. R. WILLIAMS	554
On the Trigonometric Developments of Certain Doubly Periodic Functions of the Second Kind. By M. A. BASOCO	560
Quadratic Partitions: Paper III. By E. T. BELL	569
A Logical Expansion in Mathematics. By HASSLER WHITNEY	570
On Angles in Certain Metric Spaces. By W. A. WILSON	580
Relation of Whitehead and Russell's Theory of Deduction to the Boolean Logic of Propositions. By B. A. BERNSTEIN	589
On Algebraic Equations having only Real Roots. By W. E. ROTH	594

For official Communications and Notices, see the inside of the back cover.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN